UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 6 June, 2014 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

*Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles labelled **A, B, . . . , H** according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

Let V denote the vector space of all real polynomials of degree at most 2. Show that

$$(f,g) \;=\; \int_{-1}^{1} f(x)g(x)\,dx$$

defines an inner product on V.

Find an orthonormal basis for V.

2E Groups, Rings and Modules

Let G be the abelian group generated by elements a, b and c subject to the relations: $3a+6b+3c=0$, $9b+9c=0$ and $-3a+3b+6c=0$. Express G as a product of cyclic groups. Hence determine the number of elements of G of order 3.

3F Analysis II

Define a *contraction mapping* and state the contraction mapping theorem.

Let $C[0,1]$ be the space of continuous real-valued functions on $[0,1]$ endowed with the uniform norm. Show that the map $A : C[0,1] \to C[0,1]$ defined by

$$Af(x) = \int_0^x f(t)dt$$

is not a contraction mapping, but that $A \circ A$ is.

4G Complex Analysis

Let f be an entire function. State Cauchy's Integral Formula, relating the nth derivative of f at a point z with the values of f on a circle around z.

State Liouville's Theorem, and deduce it from Cauchy's Integral Formula.

Let f be an entire function, and suppose that for some k we have that $|f(z)| \leqslant |z|^k$ for all z. Prove that f is a polynomial.

5D Methods

Consider the ordinary differential equation

$$\frac{d^2\psi}{dz^2} - \left[\frac{15k^2}{4(k|z|+1)^2} - 3k\delta(z)\right]\psi = 0\,, \qquad (\dagger)$$

where k is a positive constant and δ denotes the Dirac delta function. Physically relevant solutions for ψ are bounded over the entire range $z \in \mathbb{R}$.

(i) Find piecewise bounded solutions to this differential equations in the ranges $z > 0$ and $z < 0$, respectively. [*Hint: The equation* $\frac{d^2y}{dx^2} - \frac{c}{x^2}y = 0$ *for a constant* c *may be solved using the Ansatz* $y = x^\alpha$.]

(ii) Derive a matching condition at $z = 0$ by integrating ($\dagger$) over the interval $(-\epsilon, \epsilon)$ with $\epsilon \to 0$ and use this condition together with the requirement that ψ be continuous at $z = 0$ to determine the solution over the entire range $z \in \mathbb{R}$.

6A Quantum Mechanics

For some quantum mechanical observable Q, prove that its uncertainty (ΔQ) satisfies

$$(\Delta Q)^2 = \langle Q^2 \rangle - \langle Q \rangle^2 .$$

A quantum mechanical harmonic oscillator has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2},$$

where $m > 0$. Show that (in a stationary state of energy E)

$$E \geqslant \frac{(\Delta p)^2}{2m} + \frac{m\omega^2(\Delta x)^2}{2}.$$

Write down the Heisenberg uncertainty relation. Then, use it to show that

$$E \geqslant \frac{1}{2}\hbar\omega$$

for our stationary state.

7A Electromagnetism

A continuous wire of resistance R is wound around a very long right circular cylinder of radius a, and length l (long enough so that end effects can be ignored). There are $N \gg 1$ turns of wire per unit length, wound in a spiral of very small pitch. Initially, the magnetic field $\mathbf{B}$ is $\mathbf{0}$.

Both ends of the coil are attached to a battery of electromotance $\mathcal{E}_0$ at $t = 0$, which induces a current $I(t)$. Use Ampère's law to derive $\mathbf{B}$ inside and outside the cylinder when the displacement current may be neglected. Write the self-inductance of the coil L in terms of the quantities given above. Using Ohm's law and Faraday's law of induction, find $I(t)$ explicitly in terms of $\mathcal{E}_0$, R, L and t.

8C Numerical Analysis

Consider the quadrature given by

$$\int_0^\pi w(x)f(x)dx \approx \sum_{k=1}^{\nu} b_k f(c_k)$$

for $\nu \in \mathbb{N}$, disjoint $c_k \in (0,\pi)$ and $w > 0$. Show that it is not possible to make this quadrature exact for all polynomials of order 2ν.

For the case that $\nu = 2$ and $w(x) = \sin x$, by considering orthogonal polynomials find suitable b_k and c_k that make the quadrature exact on cubic polynomials.

[*Hint:* $\int_0^\pi x^2 \sin x \, dx = \pi^2 - 4$ *and* $\int_0^\pi x^3 \sin x \, dx = \pi^3 - 6\pi$.]

9H Markov Chains

Let $(X_n : n \geqslant 0)$ be a homogeneous Markov chain with state space S and transition matrix $P = (p_{i,j} : i, j \in S)$.

(a) Let $W_n = X_{2n}$, $n = 0, 1, 2, \ldots$. Show that $(W_n : n \geqslant 0)$ is a Markov chain and give its transition matrix. If $\lambda_i = \mathbb{P}(X_0 = i)$, $i \in S$, find $\mathbb{P}(W_1 = 0)$ in terms of the λ_i and the $p_{i,j}$.

[Results from the course may be quoted without proof, provided they are clearly stated.]

(b) Suppose that $S = \{-1, 0, 1\}$, $p_{0,1} = p_{-1,-1} = 0$ and $p_{-1,0} \neq p_{1,0}$. Let $Y_n = |X_n|$, $n = 0, 1, 2, \ldots$. In terms of the $p_{i,j}$, find

(i) $\mathbb{P}(Y_{n+1} = 0 \mid Y_n = 1, Y_{n-1} = 0)$ and

(ii) $\mathbb{P}(Y_{n+1} = 0 \mid Y_n = 1, Y_{n-1} = 1, Y_{n-2} = 0)$.

What can you conclude about whether or not $(Y_n : n \geqslant 0)$ is a Markov chain?

SECTION II

10G Linear Algebra

Let V be a real vector space. What is the *dual* V^* of V? If $e_1, \ldots, e_n$ is a basis for V, define the *dual basis* $e_1^*, \ldots, e_n^*$ for V^*, and show that it is indeed a basis for V^*.

[No result about dimensions of dual spaces may be assumed.]

For a subspace U of V, what is the *annihilator* of U? If V is n-dimensional, how does the dimension of the annihilator of U relate to the dimension of U?

Let $\alpha : V \to W$ be a linear map between finite-dimensional real vector spaces. What is the *dual map* α^*? Explain why the rank of α^* is equal to the rank of α. Prove that the kernel of α^* is the annihilator of the image of α, and also that the image of α^* is the annihilator of the kernel of α.

[Results about the matrices representing a map and its dual may be used without proof, provided they are stated clearly.]

Now let V be the vector space of all real polynomials, and define elements $L_0, L_1, \ldots$ of V^* by setting $L_i(p)$ to be the coefficient of X^i in p (for each $p \in V$). Do the L_i form a basis for V^*?

11E Groups, Rings and Modules

(a) Consider the four following types of rings: Principal Ideal Domains, Integral Domains, Fields, and Unique Factorisation Domains. Arrange them in the form $A \implies B \implies C \implies D$ (where $A \implies B$ means if a ring is of type A then it is of type B).

Prove that these implications hold. [You may assume that irreducibles in a Principal Ideal Domain are prime.] Provide examples, with brief justification, to show that these implications cannot be reversed.

(b) Let R be a ring with ideals I and J satisfying $I \subseteq J$. Define K to be the set $\{r \in R : rJ \subseteq I\}$. Prove that K is an ideal of R. If J and K are principal, prove that I is principal.

12F Analysis II

Let $U \subset \mathbb{R}^2$ be an open set. Define what it means for a function $f : U \to \mathbb{R}$ to be *differentiable* at a point $(x_0, y_0) \in U$.

Prove that if the partial derivatives $D_1 f$ and $D_2 f$ exist on U and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

If f is differentiable on U must $D_1 f$, $D_2 f$ be continuous at (x_0, y_0)? Give a proof or counterexample as appropriate.

The function $h : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$h(x, y) = xy \sin(1/x) \quad \text{for } x \neq 0, \qquad h(0, y) = 0.$$

Determine all the points (x, y) at which h is differentiable.

13E Metric and Topological Spaces

Explain what it means for a metric space to be *complete*.

Let X be a metric space. We say the subsets A_i of X, with $i \in \mathbb{N}$, form a *descending sequence* in X if $A_1 \supset A_2 \supset A_3 \supset \cdots$.

Prove that the metric space X is complete if and only if any descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of X, such that the diameters of the subsets A_i converge to zero, has an intersection $\bigcap_{i=1}^{\infty} A_i$ that is non-empty.

[Recall that the diameter $\operatorname{diam}(S)$ of a set S is the supremum of the set $\{d(x, y) : x, y \in S\}$.]

Give examples of

(i) a metric space X, and a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of X, with $\operatorname{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

(ii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty sets in $\mathbb{R}$ with $\operatorname{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

(iii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed sets in $\mathbb{R}$ with $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

14B Complex Methods

Find the Laplace transforms of t^n for n a positive integer and $H(t-a)$ where $a>0$ and $H(t)$ is the Heaviside step function.

Consider a semi-infinite string which is initially at rest and is fixed at one end. The string can support wave-like motions, and for $t>0$ it is allowed to fall under gravity. Therefore the deflection $y(x,t)$ from its initial location satisfies

$$\frac{\partial^2}{\partial t^2}y = c^2\frac{\partial^2}{\partial x^2}y + g \quad \text{for} \quad x>0,\ t>0$$

with

$$y(0,t) = y(x,0) = \frac{\partial}{\partial t}y(x,0) = 0 \quad \text{and} \quad y(x,t) \to \frac{gt^2}{2} \text{ as } x \to \infty,$$

where g is a constant. Use Laplace transforms to find $y(x,t)$.

[The convolution theorem for Laplace transforms may be quoted without proof.]

15F Geometry

Define an embedded parametrised surface in $\mathbb{R}^3$. What is the Riemannian metric induced by a parametrisation? State, in terms of the Riemannian metric, the equations defining a geodesic curve $\gamma : (0,1) \to S$, assuming that γ is parametrised by arc-length.

Let S be a conical surface

$$S = \{(x,y,z) \in \mathbb{R}^3 :\ 3(x^2+y^2) = z^2,\ \ z>0\}.$$

Using an appropriate smooth parametrisation, or otherwise, prove that S is locally isometric to the Euclidean plane. Show that any two points on S can be joined by a geodesic. Is this geodesic always unique (up to a reparametrisation)? Justify your answer.

[The expression for the Euclidean metric in polar coordinates on $\mathbb{R}^2$ may be used without proof.]

16C Variational Principles

Consider the integral

$$I = \int f(y, y')dx.$$

Show that if f satisfies the Euler–Lagrange equation, then

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}.$$

An axisymmetric soap film $y(x)$ is formed between two circular wires at $x = \pm l$. The wires both have radius r. Show that the shape that minimises the surface area takes the form

$$y(x) = k\cosh\frac{x}{k}.$$

Show that there exist two possible k that satisfy the boundary conditions for r/l sufficiently large.

Show that for these solutions the second variation is given by

$$\delta^2 I = \pi\int_{-l}^{+l}\left(k\eta'^2 - \frac{1}{k}\eta^2\right)\operatorname{sech}^2\left(\frac{x}{k}\right)dx$$

where η is an axisymmetric perturbation with $\eta(\pm l) = 0$.

17D Methods

Let $f(x)$ be a complex-valued function defined on the interval $[-L, L]$ and periodically extended to $x \in \mathbb{R}$.

(i) Express $f(x)$ as a complex Fourier series with coefficients c_n, $n \in \mathbb{Z}$. How are the coefficients c_n obtained from $f(x)$?

(ii) State Parseval's theorem for complex Fourier series.

(iii) Consider the function $f(x) = \cos(\alpha x)$ on the interval $[-\pi, \pi]$ and periodically extended to $x \in \mathbb{R}$ for a complex but non-integer constant α. Calculate the complex Fourier series of $f(x)$.

(iv) Prove the formula

$$\sum_{n=1}^{\infty}\frac{1}{n^2-\alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha\tan(\alpha\pi)}.$$

(v) Now consider the case where α is a real, non-integer constant. Use Parseval's theorem to obtain a formula for

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n^2-\alpha^2)^2}.$$

What value do you obtain for this series for $\alpha = 5/2$?

18B Fluid Dynamics

Consider a layer of fluid of constant density ρ and equilibrium depth h_0 in a rotating frame of reference, rotating at constant angular velocity Ω about the vertical z-axis. The equations of motion are

$$
\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y}, \\
0 &= -\frac{\partial p}{\partial z} - \rho g,
\end{aligned}
$$

where p is the fluid pressure, u and v are the fluid velocities in the x-direction and y-direction respectively, $f = 2\Omega$, and g is the constant acceleration due to gravity. You may also assume that the horizontal extent of the layer is sufficiently large so that the layer may be considered to be shallow, such that vertical velocities may be neglected.

By considering mass conservation, show that the depth $h(x, y, t)$ of the layer satisfies

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0.$$

Now assume that $h = h_0 + \eta(x, y, t)$, where $|\eta| \ll h_0$. Show that the (linearised) potential vorticity $\mathbf{Q} = Q\hat{\mathbf{z}}$, defined by

$$Q = \zeta - \eta\frac{f}{h_0}, \text{ where } \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

and $\hat{\mathbf{z}}$ is the unit vector in the vertical z-direction, is a constant in time, i.e. $Q = Q_0(x, y)$.

When $Q_0 = 0$ everywhere, establish that the surface perturbation η satisfies

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) + f^2\eta = 0,$$

and show that this equation has wave-like solutions $\eta = \eta_0 \cos[k(x - ct)]$ when c and k are related through a dispersion relation to be determined. Show that, to leading order, the trajectories of fluid particles for these waves are ellipses. Assuming that $\eta_0 > 0$, $k > 0$, $c > 0$ and $f > 0$, sketch the fluid velocity when $k(x - ct) = n\pi/2$ for $n = 0, 1, 2, 3$.

19H Statistics

Consider a linear model

$$\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{$\dagger$}$$

where X is a known $n \times p$ matrix, $\boldsymbol{\beta}$ is a $p \times 1$ ($p < n$) vector of unknown parameters and $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of independent $N(0, \sigma^2)$ random variables with σ^2 unknown. Assume that X has full rank p. Find the least squares estimator $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ and derive its distribution. Define the residual sum of squares RSS and write down an unbiased estimator $\hat{\sigma}^2$ of σ^2.

Suppose that $V_i = a + bu_i + \delta_i$ and $Z_i = c + dw_i + \eta_i$, for $i = 1, \ldots, m$, where u_i and w_i are known with $\sum_{i=1}^m u_i = \sum_{i=1}^m w_i = 0$, and $\delta_1, \ldots, \delta_m, \eta_1, \ldots, \eta_m$ are independent $N(0, \sigma^2)$ random variables. Assume that at least two of the u_i are distinct and at least two of the w_i are distinct. Show that $\mathbf{Y} = (V_1, \ldots, V_m, Z_1, \ldots, Z_m)^T$ (where T denotes transpose) may be written as in ($\dagger$) and identify X and $\boldsymbol{\beta}$. Find $\hat{\boldsymbol{\beta}}$ in terms of the V_i, Z_i, u_i and w_i. Find the distribution of $\hat{b} - \hat{d}$ and derive a 95% confidence interval for $b - d$.

[*Hint: You may assume that* $\frac{RSS}{\sigma^2}$ *has a* χ^2_{n-p} *distribution, and that* $\hat{\beta}$ *and the residual sum of squares are independent. Properties of* χ^2 *distributions may be used without proof.*]

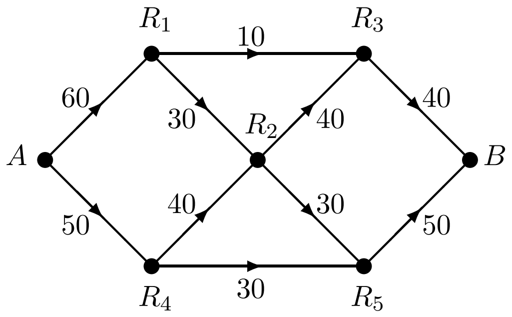
20H Optimization

Consider a network with a single source and a single sink, where all the edge capacities are finite. Write down the maximum flow problem, and state the max-flow min-cut theorem.

Describe the Ford–Fulkerson algorithm. If all edge capacities are integers, explain why, starting from a suitable initial flow, the algorithm is guaranteed to end after a finite number of iterations.

The graph in the diagram below represents a one-way road network taking traffic from point A to point B via five roundabouts R_i, $i = 1, \ldots, 5$. The capacity of each road is shown on the diagram in terms of vehicles per minute. Assuming that all roundabouts can deal with arbitrary amounts of flow of traffic, find the maximum flow of traffic (in vehicles per minute) through this network of roads. Show that this flow is indeed optimal.

After a heavy storm, roundabout R_2 is flooded and only able to deal with at most 20 vehicles per minute. Find a suitable new network for the situation after the storm. Apply the Ford–Fulkerson algorithm to the new network, starting with the zero flow and explaining each step, to determine the maximum flow and the associated flows on each road.



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