UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday, 3 June, 2014 9:00 am to 12:00 pm

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

State and prove the Rank–Nullity Theorem.

Let α be a linear map from $\mathbb{R}^5$ to $\mathbb{R}^3$. What are the possible dimensions of the kernel of α? Justify your answer.

2E Groups, Rings and Modules

List the conjugacy classes of A_6 and determine their sizes. Hence prove that A_6 is simple.

3F Analysis II

Define what is meant by a *uniformly continuous* function on a set $E \subset \mathbb{R}$.

If f and g are uniformly continuous functions on $\mathbb{R}$, is the (pointwise) product fg necessarily uniformly continuous on $\mathbb{R}$?

Is a uniformly continuous function on $(0,1)$ necessarily bounded?

Is $\cos(1/x)$ uniformly continuous on $(0,1)$?

Justify your answers.

4E Metric and Topological Spaces

Let X and Y be topological spaces. What does it mean to say that a function $f : X \to Y$ is *continuous*?

Are the following statements true or false? Give proofs or counterexamples.

(i) Every continuous function $f : X \to Y$ is an open map, i.e. if U is open in X then $f(U)$ is open in Y.

(ii) If $f : X \to Y$ is continuous and bijective then f is a homeomorphism.

(iii) If $f : X \to Y$ is continuous, open and bijective then f is a homeomorphism.

5D Methods

(i) Calculate the Fourier series for the periodic extension on $\mathbb{R}$ of the function

$$f(x) = x(1-x)\,,$$

defined on the interval $[0,1)$.

(ii) Explain why the Fourier series for the periodic extension of $f'(x)$ can be obtained by term-by-term differentiation of the series for $f(x)$.

(iii) Let $G(x)$ be the Fourier series for the periodic extension of $f'(x)$. Determine the value of $G(0)$ and explain briefly how it is related to the values of f'.

6A Electromagnetism

Starting from Maxwell's equations, deduce that

$$\frac{d\Phi}{dt} = -\mathcal{E},$$

for a moving circuit C, where Φ is the flux of $\mathbf{B}$ through the circuit and where the electromotive force $\mathcal{E}$ is defined to be

$$\mathcal{E} = \oint_C (\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot \mathbf{dr}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{r})$ denotes the velocity of a point $\mathbf{r}$ on C.

[*Hint: Consider the closed surface consisting of the surface $S(t)$ bounded by $C(t)$, the surface $S(t+\delta t)$ bounded by $C(t+\delta t)$ and the surface S' stretching from $C(t)$ to $C(t+\delta t)$. Show that the flux of* $\mathbf{B}$ *through S' is* $-\delta t \oint_C \mathbf{B}\cdot(\mathbf{v}\times\mathbf{dr})$.]

7B Fluid Dynamics

Consider the steady two-dimensional fluid velocity field

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \epsilon & -\gamma \\ \gamma & -\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\epsilon \geqslant 0$ and $\gamma \geqslant 0$. Show that the fluid is incompressible. The streamfunction ψ is defined by $\mathbf{u} = \boldsymbol{\nabla} \times \boldsymbol{\Psi}$, where $\boldsymbol{\Psi} = (0, 0, \psi)$. Show that ψ is given by

$$\psi = \epsilon xy - \frac{\gamma}{2}(x^2 + y^2).$$

Hence show that the streamlines are defined by

$$(\epsilon - \gamma)(x + y)^2 - (\epsilon + \gamma)(x - y)^2 = C,$$

for C a constant. For each of the three cases below, sketch the streamlines and briefly describe the flow.

(i) $\epsilon = 1$, $\gamma = 0$,

(ii) $\epsilon = 0$, $\gamma = 1$,

(iii) $\epsilon = 1$, $\gamma = 1$.

8H Statistics

There are 100 patients taking part in a trial of a new surgical procedure for a particular medical condition. Of these, 50 patients are randomly selected to receive the new procedure and the remaining 50 receive the old procedure. Six months later, a doctor assesses whether or not each patient has fully recovered. The results are shown below:

	Fully recovered	Not fully recovered
Old procedure	25	25
New procedure	31	19

The doctor is interested in whether there is a difference in full recovery rates for patients receiving the two procedures. Carry out an appropriate 5% significance level test, stating your hypotheses carefully. [You do not need to derive the test.] What conclusion should be reported to the doctor?

[*Hint: Let* $\chi_k^2(\alpha)$ *denote the upper* 100α *percentage point of a* χ_k^2 *distribution. Then*

$$\chi_1^2(0.05) = 3.84,\ \chi_2^2(0.05) = 5.99,\ \chi_3^2(0.05) = 7.82,\ \chi_4^2(0.05) = 9.49.]$$

9H Optimization

Explain what is meant by a two-player zero-sum game with $m \times n$ pay-off matrix $P = (p_{ij})$, and state the optimal strategies for each player.

Find these optimal strategies when

$$P = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}.$$

SECTION II

10G Linear Algebra

Define the *determinant* of an $n \times n$ complex matrix A. Explain, with justification, how the determinant of A changes when we perform row and column operations on A.

Let A, B, C be complex $n \times n$ matrices. Prove the following statements.

(i) $\det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det A \ \det B$.

(ii) $\det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det(A + iB) \ \det(A - iB)$.

11E Groups, Rings and Modules

Prove that every finite integral domain is a field.

Let F be a field and f an irreducible polynomial in the polynomial ring $F[X]$. Prove that $F[X]/(f)$ is a field, where (f) denotes the ideal generated by f.

Hence construct a field of 4 elements, and write down its multiplication table.

Construct a field of order 9.

12F Analysis II

Let $X,\ Y$ be subsets of $\mathbb{R}^n$ and define $X + Y = \{x + y : x \in X,\ y \in Y\}$. For each of the following statements give a proof or a counterexample (with justification) as appropriate.

(i) If each of $X,\ Y$ is bounded and closed, then $X + Y$ is bounded and closed.

(ii) If X is bounded and closed and Y is closed, then $X + Y$ is closed.

(iii) If $X,\ Y$ are both closed, then $X + Y$ is closed.

(iv) If X is open and Y is closed, then $X + Y$ is open.

[The Bolzano–Weierstrass theorem in $\mathbb{R}^n$ may be assumed without proof.]

13B Complex Analysis or Complex Methods

By considering a rectangular contour, show that for $0 < a < 1$ we have

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx = \frac{\pi}{\sin \pi a}.$$

Hence evaluate

$$\int_0^{\infty} \frac{dt}{t^{5/6}(1+t)}.$$

14F Geometry

Let $H = \{x + iy : x, y \in \mathbb{R},\ y > 0\} \subset \mathbb{C}$ be the upper half-plane with a hyperbolic metric $g = \frac{dx^2+dy^2}{y^2}$. Prove that every hyperbolic circle C in H is also a Euclidean circle. Is the centre of C as a hyperbolic circle always the same point as the centre of C as a Euclidean circle? Give a proof or counterexample as appropriate.

Let ABC and $A'B'C'$ be two hyperbolic triangles and denote the hyperbolic lengths of their sides by a, b, c and a', b', c', respectively. Show that if $a = a'$, $b = b'$ and $c = c'$, then there is a hyperbolic isometry taking ABC to $A'B'C'$. Is there always such an isometry if instead the triangles have one angle the same and $a = a'$, $b = b'$? Justify your answer.

[Standard results on hyperbolic isometries may be assumed, provided they are clearly stated.]

15C Variational Principles

Write down the Euler–Lagrange equation for the integral

$$\int f(y, y', x) dx.$$

An ant is walking on the surface of a sphere, which is parameterised by $\theta \in [0, \pi]$ (angle from top of sphere) and $\phi \in [0, 2\pi)$ (azimuthal angle). The sphere is sticky towards the top and the bottom and so the ant's speed is proportional to $\sin \theta$. Show that the ant's fastest route between two points will be of the form

$$\sinh(A\phi + B) = \cot \theta$$

for some constants A and B. [A, B need not be determined.]

16D Methods

The Fourier transform $\tilde{f}$ of a function f is defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \,, \quad \text{so that} \;\; f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \,.$$

A Green's function $G(t, t', x, x')$ for the diffusion equation in one spatial dimension satisfies

$$\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2} = \delta(t-t')\,\delta(x-x')\,.$$

(a) By applying a Fourier transform, show that the Fourier transform $\tilde{G}$ of this Green's function and the Green's function G are

$$\begin{aligned} \tilde{G}(t, t', k, x') &= H(t-t')\, e^{-ikx'}\, e^{-Dk^2(t-t')}\,, \\ G(t, t', x, x') &= \frac{H(t-t')}{\sqrt{4\pi D(t-t')}} e^{-\frac{(x-x')^2}{4D(t-t')}}\,, \end{aligned}$$

where H is the Heaviside function. [*Hint: The Fourier transform* $\tilde{F}$ *of a Gaussian* $F(x) = \dfrac{1}{\sqrt{4\pi a}} e^{-\frac{x^2}{4a}}$, $a = \text{const}$, *is given by* $\tilde{F}(k) = e^{-ak^2}$.]

(b) The analogous result for the Green's function for the diffusion equation in two spatial dimensions is

$$G(t, t', x, x', y, y') = \frac{H(t-t')}{4\pi D(t-t')} e^{-\frac{1}{4D(t-t')}\left[(x-x')^2 + (y-y')^2\right]}\,.$$

Use this Green's function to construct a solution for $t \geqslant 0$ to the diffusion equation

$$\frac{\partial \Psi}{\partial t} - D\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right) = p(t)\,\delta(x)\,\delta(y)\,,$$

with the initial condition $\Psi(0, x, y) = 0$.

Now set

$$p(t) = \begin{cases} p_0 = \text{const} & \text{for} \quad 0 \leqslant t \leqslant t_0 \\ 0 & \text{for} \quad t > t_0 \end{cases}$$

Find the solution $\Psi(t, x, y)$ for $t > t_0$ in terms of the exponential integral defined by

$$Ei(-\eta) = -\int_{\eta}^{\infty} \frac{e^{-\lambda}}{\lambda} d\lambda \,.$$

Use the approximation $Ei(-\eta) \approx \ln \eta + C$, valid for $\eta \ll 1$, to simplify this solution $\Psi(t, x, y)$. Here $C \approx 0.577$ is Euler's constant.

17A Quantum Mechanics

For an electron of mass m in a hydrogen atom, the time-independent Schrödinger equation may be written as

$$-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{2mr^2}L^2\psi-\frac{e^2}{4\pi\epsilon_0 r}\psi=E\psi.$$

Consider normalised energy eigenstates of the form

$$\psi_{lm}(r,\ \theta,\ \phi)=R(r)Y_{lm}(\theta,\ \phi)$$

where Y_{lm} are orbital angular momentum eigenstates:

$$L^2Y_{lm}=\hbar^2 l(l+1)Y_{lm},\qquad L_3Y_{lm}=\hbar mY_{lm},$$

where $l=1,\ 2,\ \ldots$ and $m=0,\ \pm1,\ \pm2,\ \ldots\pm l$. The Y_{lm} functions are normalised with $\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}|Y_{lm}|^2\ \sin\theta\ d\theta\ d\phi=1$.

(i) Write down the resulting equation satisfied by $R(r)$, for fixed l. Show that it has solutions of the form

$$R(r)=Ar^l\exp\left(-\frac{r}{a(l+1)}\right),$$

where a is a constant which you should determine. Show that

$$E=-\frac{e^2}{D\pi\epsilon_0 a},$$

where D is an integer which you should find (in terms of l). Also, show that

$$|A|^2=\frac{2^{2l+3}}{a^F G!(l+1)^H},$$

where F, G and H are integers that you should find in terms of l.

(ii) Given the radius of the proton $r_p \ll a$, show that the probability of the electron being found within the proton is approximately

$$\frac{2^{2l+3}}{C}\left(\frac{r_p}{a}\right)^{2l+3}\left[1+\mathcal{O}\left(\frac{r_p}{a}\right)\right],$$

finding the integer C in terms of l.

[You may assume that $\int_0^\infty t^l e^{-t}dt=l!$.]

18A Electromagnetism

What is the relationship between the electric field $\mathbf{E}$ and the charge per unit area σ on the surface of a perfect conductor?

Consider a charge distribution $\rho(\mathbf{r})$ distributed with potential $\phi(\mathbf{r})$ over a finite volume V within which there is a set of perfect conductors with charges Q_i, each at a potential ϕ_i (normalised such that the potential at infinity is zero). Using Maxwell's equations and the divergence theorem, derive a relationship between the electrostatic energy W and a volume integral of an explicit function of the electric field $\mathbf{E}$, where

$$W = \frac{1}{2}\int_V \rho\phi \; d\tau + \frac{1}{2}\sum_i Q_i\phi_i \,.$$

Consider N concentric perfectly conducting spherical shells. Shell n has radius r_n (where $r_n > r_{n-1}$) and charge q for $n = 1$, and charge $2(-1)^{(n+1)}q$ for $n > 1$. Show that

$$W \propto \frac{1}{r_1},$$

and determine the constant of proportionality.

19C Numerical Analysis

A linear functional acting on $f \in C^{k+1}[a,b]$ is approximated using a linear scheme $L(f)$. The approximation is exact when f is a polynomial of degree k. The error is given by $\lambda(f)$. Starting from the Taylor formula for $f(x)$ with an integral remainder term, show that the error can be written in the form

$$\lambda(f) = \frac{1}{k!}\int_a^b K(\theta) f^{(k+1)}(\theta) d\theta$$

subject to a condition on λ that you should specify. Give an expression for $K(\theta)$.

Find c_0, c_1 and c_3 such that the approximation scheme

$$f''(2) \approx c_0 f(0) + c_1 f(1) + c_3 f(3)$$

is exact for all f that are polynomials of degree 2. Assuming $f \in C^3[0,3]$, apply the Peano kernel theorem to the error. Find and sketch $K(\theta)$ for $k = 2$.

Find the minimum values for the constants r and s for which

$$|\lambda(f)| \leqslant r\|f^{(3)}\|_1 \quad \text{and} \quad |\lambda(f)| \leqslant s\|f^{(3)}\|_\infty$$

and show explicitly that both error bounds hold for $f(x) = x^3$.

20H Markov Chains

Let $(X_n : n \geqslant 0)$ be a homogeneous Markov chain with state space S and transition matrix $P = (p_{i,j} : i, j \in S)$. For $A \subseteq S$, let

$$H^A = \inf\{n \geqslant 0 : X_n \in A\} \text{ and } h_i^A = \mathbb{P}(H^A < \infty \mid X_0 = i),\ i \in S.$$

Prove that $h^A = (h_i^A : i \in S)$ is the minimal non-negative solution to the equations

$$h_i^A = \begin{cases} 1 & \text{for } i \in A \\ \sum_{j\in S} p_{i,j} h_j^A & \text{otherwise.} \end{cases}$$

Three people A, B and C play a series of two-player games. In the first game, two people play and the third person sits out. Any subsequent game is played between the winner of the previous game and the person sitting out the previous game. The overall winner of the series is the first person to win two consecutive games. The players are evenly matched so that in any game each of the two players has probability $\frac{1}{2}$ of winning the game, independently of all other games. For $n = 1, 2, \ldots$, let X_n be the ordered pair consisting of the winners of games n and $n+1$. Thus the state space is $\{AA, AB, AC, BA, BB, BC, CA, CB, CC\}$, and, for example, $X_1 = AC$ if A wins the first game and C wins the second.

The first game is between A and B. Treating AA, BB and CC as absorbing states, or otherwise, find the probability of winning the series for each of the three players.

END OF PAPER