UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 30 May, 2014 1:30 pm to 4:30 pm

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

State and prove the Steinitz Exchange Lemma. Use it to prove that, in a finite-dimensional vector space: any two bases have the same size, and every linearly independent set extends to a basis.

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Is $e_1 + e_2,\ e_2 + e_3,\ e_3 + e_1$ a basis for $\mathbb{R}^3$? Is $e_1 + e_2,\ e_2 + e_3,\ e_3 + e_4,\ e_4 + e_1$ a basis for $\mathbb{R}^4$? Justify your answers.

2B Complex Analysis or Complex Methods

Let $f(z)$ be an analytic/holomorphic function defined on an open set D, and let $z_0 \in D$ be a point such that $f'(z_0) \neq 0$. Show that the transformation $w = f(z)$ preserves the angle between smooth curves intersecting at z_0. Find such a transformation $w = f(z)$ that maps the second quadrant of the unit disc (i.e. $|z| < 1$, $\pi/2 < \arg(z) < \pi$) to the region in the first quadrant of the complex plane where $|w| > 1$ (i.e. the region in the first quadrant *outside* the unit circle).

3F Geometry

Determine the second fundamental form of a surface in $\mathbb{R}^3$ defined by the parametrisation

$$\sigma(u, v) = \Big((a + b\cos u)\cos v,\ (a + b\cos u)\sin v,\ b\sin u\Big),$$

for $0 < u < 2\pi,\ 0 < v < 2\pi$, with some fixed $a > b > 0$. Show that the Gaussian curvature $K(u, v)$ of this surface takes both positive and negative values.

4C Variational Principles

Define the Legendre transform $f^*(\mathbf{p})$ of a function $f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$.

Show that for $g(\mathbf{x}) = \lambda f(\mathbf{x} - \mathbf{x}_0) - \mu$,

$$g^*(\mathbf{p}) = \lambda f^*\left(\frac{\mathbf{p}}{\lambda}\right) + \mathbf{p}^{\mathbf{T}}\mathbf{x}_0 + \mu.$$

Show that for $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\,\mathbf{x}$ where $\mathbf{A}$ is a real, symmetric, invertible matrix with positive eigenvalues,

$$f^*(\mathbf{p}) = \tfrac{1}{2}\mathbf{p}^{\mathbf{T}}\mathbf{A}^{-\mathbf{1}}\,\mathbf{p}.$$

5B Fluid Dynamics

Constant density viscous fluid with dynamic viscosity μ flows in a two-dimensional horizontal channel of depth h. There is a constant pressure gradient $G > 0$ in the horizontal $x-$direction. The upper horizontal boundary at $y = h$ is driven at constant horizontal speed $U > 0$, with the lower boundary being held at rest. Show that the steady fluid velocity u in the $x-$direction is

$$u = \frac{-G}{2\mu} y(h-y) + \frac{Uy}{h}.$$

Show that it is possible to have $du/dy < 0$ at some point in the flow for sufficiently large pressure gradient. Derive a relationship between G and U so that there is no net volume flux along the channel. For the flow with no net volume flux, sketch the velocity profile.

6C Numerical Analysis

(i) A general multistep method for the numerical approximation to the scalar differential equation $y' = f(t,y)$ is given by

$$\sum_{\ell=0}^{s} \rho_\ell \, y_{n+\ell} = h \sum_{\ell=0}^{s} \sigma_\ell f_{n+\ell}, \qquad n = 0, 1, \ldots$$

where $f_{n+\ell} = f(t_{n+\ell}, y_{n+\ell})$. Show that this method is of order $p \geqslant 1$ if and only if

$$\rho(\mathrm{e}^z) - z\sigma(\mathrm{e}^z) = \mathcal{O}(z^{p+1}) \quad \text{as} \quad z \to 0$$

where

$$\rho(w) = \sum_{\ell=0}^{s} \rho_\ell \, w^\ell \quad \text{and} \quad \sigma(w) = \sum_{\ell=0}^{s} \sigma_\ell \, w^\ell \, .$$

(ii) A particular three-step implicit method is given by

$$y_{n+3} + (a-1)y_{n+1} - a y_n = h \left(f_{n+3} + \sum_{\ell=0}^{2} \sigma_\ell f_{n+\ell} \right) .$$

where the σ_ℓ are chosen to make the method third order. [The σ_ℓ need not be found.] For what values of a is the method convergent?

7H Statistics

Consider an estimator $\hat{\theta}$ of an unknown parameter θ, and assume that $\mathbb{E}_\theta\big(\hat{\theta}^2\big) < \infty$ for all θ. Define the *bias* and *mean squared error* of $\hat{\theta}$.

Show that the mean squared error of $\hat{\theta}$ is the sum of its variance and the square of its bias.

Suppose that $X_1, \ldots, X_n$ are independent identically distributed random variables with mean θ and variance θ^2, and consider estimators of θ of the form $k\bar{X}$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

(i) Find the value of k that gives an unbiased estimator, and show that the mean squared error of this unbiased estimator is θ^2/n.

(ii) Find the range of values of k for which the mean squared error of $k\bar{X}$ is smaller than θ^2/n.

8H Optimization

State and prove the Lagrangian sufficiency theorem.

Use the Lagrangian sufficiency theorem to find the minimum of $2x_1^2 + 2x_2^2 + x_3^2$ subject to $x_1 + x_2 + x_3 = 1$ (where x_1, x_2 and x_3 are real).

SECTION II

9G Linear Algebra

Let V be an n-dimensional real vector space, and let T be an endomorphism of V. We say that T *acts* on a subspace W if $T(W) \subset W$.

(i) For any $x \in V$, show that T acts on the linear span of $\{x, T(x), T^2(x), \ldots, T^{n-1}(x)\}$.

(ii) If $\{x, T(x), T^2(x), \ldots, T^{n-1}(x)\}$ spans V, show directly (i.e. without using the Cayley–Hamilton Theorem) that T satisfies its own characteristic equation.

(iii) Suppose that T acts on a subspace W with $W \neq \{0\}$ and $W \neq V$. Let $e_1, \ldots, e_k$ be a basis for W, and extend to a basis $e_1, \ldots, e_n$ for V. Describe the matrix of T with respect to this basis.

(iv) Using (i), (ii) and (iii) and induction, give a proof of the Cayley–Hamilton Theorem.

[Simple properties of determinants may be assumed without proof.]

10E Groups, Rings and Modules

Let G be a finite group and p a prime divisor of the order of G. Give the definition of a Sylow p-subgroup of G, and state Sylow's theorems.

Let p and q be distinct primes. Prove that a group of order p^2q is not simple.

Let G be a finite group, H a normal subgroup of G and P a Sylow p-subgroup of H. Let $N_G(P)$ denote the normaliser of P in G. Prove that if $g \in G$ then there exist $k \in N_G(P)$ and $h \in H$ such that $g = kh$.

11F Analysis II

Define what it means for two norms on a real vector space V to be Lipschitz equivalent. Show that if two norms on V are Lipschitz equivalent and $F \subset V$, then F is closed in one norm if and only if F is closed in the other norm.

Show that if V is finite-dimensional, then any two norms on V are Lipschitz equivalent.

Show that $\|f\|_1 = \int_0^1 |f(x)|dx$ is a norm on the space $C[0,1]$ of continuous real-valued functions on $[0,1]$. Is the set $S = \{f \in C[0,1] : f(1/2) = 0\}$ closed in the norm $\|\cdot\|_1$?

Determine whether or not the norm $\|\cdot\|_1$ is Lipschitz equivalent to the uniform norm $\|\cdot\|_\infty$ on $C[0,1]$.

[You may assume the Bolzano–Weierstrass theorem for sequences in $\mathbb{R}^n$.]

12E Metric and Topological Spaces

Define what it means for a topological space to be *compact*. Define what it means for a topological space to be *Hausdorff*.

Prove that a compact subspace of a Hausdorff space is closed. Hence prove that if C_1 and C_2 are compact subspaces of a Hausdorff space X then $C_1 \cap C_2$ is compact.

A subset U of $\mathbb{R}$ is open in the *cocountable topology* if U is empty or its complement in $\mathbb{R}$ is countable. Is $\mathbb{R}$ Hausdorff in the cocountable topology? Which subsets of $\mathbb{R}$ are compact in the cocountable topology?

13B Complex Analysis or Complex Methods

By choice of a suitable contour show that for $a > b > 0$

$$\int_0^{2\pi} \frac{\sin^2 \theta d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left[a - \sqrt{a^2 - b^2}\right].$$

Hence evaluate

$$\int_0^1 \frac{(1-x^2)^{1/2} x^2 dx}{1 + x^2}$$

using the substitution $x = \cos(\theta/2)$.

14D Methods

(a) Legendre's differential equation may be written

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0\,, \qquad y(1) = 1\,.$$

Show that for non-negative integer n, this equation has a solution $P_n(x)$ that is a polynomial of degree n. Find P_0, P_1 and P_2 explicitly.

(b) Laplace's equation in spherical coordinates for an axisymmetric function $U(r,\theta)$ (i.e. no ϕ dependence) is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) = 0\,.$$

Use separation of variables to find the general solution for $U(r,\theta)$.

Find the solution $U(r,\theta)$ that satisfies the boundary conditions

$$U(r,\theta) \to v_0\, r\, \cos\theta \qquad \text{as } r \to \infty\,,$$

$$\frac{\partial U}{\partial r} = 0 \qquad \text{at } r = r_0\,,$$

where v_0 and r_0 are constants.

15A Quantum Mechanics

Consider a particle confined in a one-dimensional infinite potential well: $V(x) = \infty$ for $|x| \geqslant a$ and $V(x) = 0$ for $|x| < a$. The normalised stationary states are

$$\psi_n(x) = \begin{cases} \alpha_n \sin\left(\dfrac{\pi n(x+a)}{2a}\right) & \text{for} \quad |x| < a \\ 0 & \text{for} \quad |x| \geqslant a \end{cases}$$

where $n = 1, 2, \ldots$.

(i) Determine the α_n and the stationary states' energies E_n.

(ii) A state is prepared within this potential well: $\psi(x) \propto x$ for $0 < x < a$, but $\psi(x) = 0$ for $x \leqslant 0$ or $x \geqslant a$. Find an explicit expansion of $\psi(x)$ in terms of $\psi_n(x)$.

(iii) If the energy of the state is then immediately measured, show that the probability that it is *greater* than $\frac{\hbar^2\pi^2}{ma^2}$ is

$$\sum_{n=0}^{4} \frac{b_n}{\pi^n},$$

where the b_n are integers which you should find.

(iv) By considering the normalisation condition for $\psi(x)$ in terms of the expansion in $\psi_n(x)$, show that

$$\frac{\pi^2}{3} = \sum_{p=1}^{\infty} \frac{A}{p^2} + \frac{B}{(2p-1)^2}\left(1 + \frac{C(-1)^p}{(2p-1)\pi}\right)^2,$$

where A, B and C are integers which you should find.

16A Electromagnetism

The region $z < 0$ is occupied by an ideal earthed conductor and a point charge q with mass m is held above it at $(0,\ 0,\ d)$.

(i) What are the boundary conditions satisfied by the electric field $\mathbf{E}$ on the surface of the conductor?

(ii) Consider now a system without the conductor mentioned above. A point charge q with mass m is held at $(0,\ 0,\ d)$, and one of charge $-q$ is held at $(0,\ 0,\ -d)$. Show that the boundary condition on $\mathbf{E}$ at $z = 0$ is identical to the answer to (i). Explain why this represents the electric field due to the charge at $(0,\ 0,\ d)$ under the influence of the conducting boundary.

(iii) The original point charge in (i) is released with zero initial velocity. Find the time taken for the point charge to reach the plane (ignoring gravity).

[You may assume that the force on the point charge is equal to $m\, d^2\mathbf{x}/dt^2$, where $\mathbf{x}$ is the position vector of the charge, and t is time.]

17B Fluid Dynamics

Consider the purely two-dimensional steady flow of an inviscid incompressible constant density fluid in the absence of body forces. For velocity $\mathbf{u}$, the vorticity is $\nabla \times \mathbf{u} = \boldsymbol{\omega} = (0, 0, \omega)$. Show that

$$\mathbf{u} \times \boldsymbol{\omega} = \nabla \left[\frac{p}{\rho} + \frac{1}{2} |\mathbf{u}|^2 \right],$$

where p is the pressure and ρ is the fluid density. Hence show that, if ω is a constant in both space and time,

$$\frac{1}{2} |\mathbf{u}|^2 + \omega \, \psi + \frac{p}{\rho} = C,$$

where C is a constant and ψ is the streamfunction. Here, ψ is defined by $\mathbf{u} = \nabla \times \boldsymbol{\Psi}$, where $\boldsymbol{\Psi} = (0, 0, \psi)$.

Fluid in the annular region $a < r < 2a$ has constant (in both space and time) vorticity ω. The streamlines are concentric circles, with the fluid speed zero on $r = 2a$ and $V > 0$ on $r = a$. Calculate the velocity field, and hence show that

$$\omega = \frac{-2V}{3a}.$$

Deduce that the pressure difference between the outer and inner edges of the annular region is

$$\Delta p = \left(\frac{15 - 16 \ln 2}{18} \right) \rho V^2.$$

[*Hint: Note that in cylindrical polar coordinates* (r, ϕ, z), *the curl of a vector field* $\mathbf{A}(r, \phi) = [a(r, \phi), b(r, \phi), c(r, \phi)]$ *is*

$$\nabla \times \mathbf{A} = \left[\frac{1}{r} \frac{\partial c}{\partial \phi}, -\frac{\partial c}{\partial r}, \frac{1}{r} \left(\frac{\partial (rb)}{\partial r} - \frac{\partial a}{\partial \phi} \right) \right]. \quad]$$

18C Numerical Analysis

Define a Householder transformation H and show that it is an orthogonal matrix. Briefly explain how these transformations can be used for QR factorisation of an $m \times n$ matrix.

Using Householder transformations, find a QR factorisation of

$$\mathsf{A} = \begin{bmatrix} 2 & 5 & 4 \\ 2 & 5 & 1 \\ -2 & 1 & 5 \\ 2 & -1 & 16 \end{bmatrix}.$$

Using this factorisation, find the value of λ for which

$$\mathsf{A}\,x = \begin{bmatrix} 1+\lambda \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

has a unique solution $x \in \mathbb{R}^3$.

19H Statistics

Suppose that X_1, X_2, and X_3 are independent identically distributed Poisson random variables with expectation θ, so that

$$\mathbb{P}(X_i = x) = \frac{e^{-\theta}\theta^x}{x!} \quad x = 0, 1, \ldots,$$

and consider testing $H_0 : \theta = 1$ against $H_1 : \theta = \theta_1$, where θ_1 is a known value greater than 1. Show that the test with critical region $\{(x_1, x_2, x_3) : \sum_{i=1}^3 x_i > 5\}$ is a likelihood ratio test of H_0 against H_1. What is the size of this test? Write down an expression for its power.

A scientist counts the number of bird territories in n randomly selected sections of a large park. Let Y_i be the number of bird territories in the ith section, and suppose that $Y_1, \ldots, Y_n$ are independent Poisson random variables with expectations $\theta_1, \ldots, \theta_n$ respectively. Let a_i be the area of the ith section. Suppose that $n = 2m$, $a_1 = \cdots = a_m = a(> 0)$ and $a_{m+1} = \cdots = a_{2m} = 2a$. Derive the generalised likelihood ratio Λ for testing

$$H_0 : \theta_i = \lambda a_i \text{ against } H_1 : \theta_i = \begin{cases} \lambda_1 & i = 1, \ldots, m \\ \lambda_2 & i = m+1, \ldots, 2m. \end{cases}$$

What should the scientist conclude about the number of bird territories if $2\log_e(\Lambda)$ is 15.67?

[*Hint: Let $F_\theta(x)$ be $\mathbb{P}(W \leqslant x)$ where W has a Poisson distribution with expectation θ. Then*

$$F_1(3) = 0.998, \quad F_3(5) = 0.916, \quad F_3(6) = 0.966, \quad F_5(3) = 0.433\,.]$$

20H Markov Chains

Consider a homogeneous Markov chain $(X_n : n \geqslant 0)$ with state space S and transition matrix $P = (p_{i,j} : i, j \in S)$. For a state i, define the terms *aperiodic*, *positive recurrent* and *ergodic*.

Let $S = \{0, 1, 2, \ldots\}$ and suppose that for $i \geqslant 1$ we have $p_{i,i-1} = 1$ and

$$p_{0,0} = 0,\ p_{0,j} = pq^{j-1},\ j = 1, 2, \ldots,$$

where $p = 1 - q \in (0, 1)$. Show that this Markov chain is irreducible.

Let $T_0 = \inf\{n \geqslant 1 : X_n = 0\}$ be the first passage time to 0. Find $\mathbb{P}(T_0 = n \mid X_0 = 0)$ and show that state 0 is ergodic.

Find the invariant distribution π for this Markov chain. Write down:

(i) the mean recurrence time for state i, $i \geqslant 1$;

(ii) $\lim_{n\to\infty} \mathbb{P}(X_n \neq 0 \mid X_0 = 0)$.

[Results from the course may be quoted without proof, provided they are clearly stated.]

END OF PAPER