UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Wednesday, 4 June, 2014 9:00 am to 12:00 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Differential Equations

The following equation arises in the theory of elastic beams:

$$t^4 \frac{d^2u}{dt^2} + \lambda^2 u = 0, \quad \lambda > 0,\ t > 0,$$

where $u(t)$ is a real valued function.

By using the change of variables

$$t = \frac{1}{\tau}, \quad u(t) = \frac{v(\tau)}{\tau},$$

find the general solution of the above equation.

2B Differential Equations

Consider the ordinary differential equation

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0. \qquad (*)$$

State an equation to be satisfied by P and Q that ensures that equation $(*)$ is exact. In this case, express the general solution of equation $(*)$ in terms of a function $F(x,y)$ which should be defined in terms of P and Q.

Consider the equation

$$\frac{dy}{dx} = -\frac{4x+3y}{3x+3y^2},$$

satisfying the boundary condition $y(1) = 2$. Find an explicit relation between y and x.

3F Probability

Consider a particle situated at the origin $(0,0)$ of $\mathbb{R}^2$. At successive times a direction is chosen independently by picking an angle uniformly at random in the interval $[0, 2\pi]$, and the particle then moves an Euclidean unit length in this direction. Find the expected squared Euclidean distance of the particle from the origin after n such movements.

4F Probability

Consider independent discrete random variables $X_1, \ldots, X_n$ and assume $E[X_i]$ exists for all $i = 1, \ldots, n$.

Show that

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i].$$

If the $X_1, \ldots, X_n$ are also positive, show that

$$\prod_{i=1}^{n} \sum_{m=0}^{\infty} P(X_i > m) = \sum_{m=0}^{\infty} P\left(\prod_{i=1}^{n} X_i > m\right).$$

SECTION II

5B Differential Equations

Use the transformation

$$y(t) = \frac{1}{cx(t)} \frac{dx(t)}{dt},$$

where c is a constant, to map the Ricatti equation

$$\frac{dy}{dt} + cy^2 + a(t)y + b(t) = 0, \quad t > 0,$$

to a linear equation.

Using the above result, as well as the change of variables $\tau = \ln t$, solve the boundary value problem

$$\frac{dy}{dt} + y^2 + \frac{y}{t} - \frac{\lambda^2}{t^2} = 0, \quad t > 0,$$

$$y(1) = 2\lambda,$$

where λ is a positive constant. What is the value of $t > 0$ for which the solution is singular?

6B Differential Equations

The so-called "shallow water theory" is characterised by the equations

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\big[(h+\zeta)u\big] = 0,$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \zeta}{\partial x} = 0,$$

where g denotes the gravitational constant, the constant h denotes the undisturbed depth of the water, $u(x,t)$ denotes the speed in the x-direction, and $\zeta(x,t)$ denotes the elevation of the water.

(i) Assuming that $|u|$ and $|\zeta|$ and their gradients are small in some appropriate dimensional considerations, show that ζ satisfies the wave equation

$$\frac{\partial^2 \zeta}{\partial t^2} = c^2\,\frac{\partial^2 \zeta}{\partial x^2}, \qquad (*)$$

where the constant c should be determined in terms of h and g.

(ii) Using the change of variables

$$\xi = x + ct, \quad \eta = x - ct,$$

show that the general solution of $(*)$ satisfying the initial conditions

$$\zeta(x,0) = u_0(x), \quad \frac{\partial \zeta}{\partial t}(x,0) = v_0(x),$$

is given by

$$\zeta(x,t) = f(x+ct) + g(x-ct),$$

where

$$\frac{df(x)}{dx} = \frac{1}{2}\left[\frac{du_0(x)}{dx} + \frac{1}{c}v_0(x)\right],$$
$$\frac{dg(x)}{dx} = \frac{1}{2}\left[\frac{du_0(x)}{dx} - \frac{1}{c}v_0(x)\right].$$

Simplify the above to find ζ in terms of u_0 and v_0.

(iii) Find $\zeta(x,t)$ in the particular case that

$$u_0(x) = H(x+1) - H(x-1), \quad v_0(x) = 0, \quad -\infty < x < \infty,$$

where $H(\cdot)$ denotes the Heaviside step function.

Describe in words this solution.

7B Differential Equations

(a) Let $y_1(x)$ be a solution of the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

Assuming that the second linearly independent solution takes the form $y_2(x) = v(x)y_1(x)$, derive an ordinary differential equation for $v(x)$.

(b) Consider the equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \quad -1 < x < 1.$$

By inspection or otherwise, find an explicit solution of this equation. Use the result in (a) to find the solution $y(x)$ satisfying the conditions

$$y(0) = \frac{dy}{dx}(0) = 1.$$

8B Differential Equations

Consider the damped pendulum equation

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \sin\theta = 0, \qquad (*)$$

where c is a positive constant. The energy E, which is the sum of the kinetic energy and the potential energy, is defined by

$$E(t) = \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + 1 - \cos\theta.$$

(i) Verify that $E(t)$ is a decreasing function.

(ii) Assuming that θ is sufficiently small, so that terms of order θ^3 can be neglected, find an approximation for the general solution of $(*)$ in terms of two arbitrary constants. Discuss the dependence of this approximate solution on c.

(iii) By rewriting $(*)$ as a system of equations for $x(t) = \theta$ and $y(t) = \dot{\theta}$, find all stationary points of $(*)$ and discuss their nature for all c, except $c = 2$.

(iv) Draw the phase plane curves for the particular case $c = 1$.

9F Probability

State the axioms of probability.

State and prove Boole's inequality.

Suppose you toss a sequence of coins, the i-th of which comes up heads with probability p_i, where $\sum_{i=1}^{\infty} p_i < \infty$. Calculate the probability of the event that infinitely many heads occur.

Suppose you repeatedly and independently roll a pair of fair dice and each time record the sum of the dice. What is the probability that an outcome of 5 appears before an outcome of 7? Justify your answer.

10F Probability

Define what it means for a random variable X to have a Poisson distribution, and find its moment generating function.

Suppose X, Y are independent Poisson random variables with parameters λ, μ. Find the distribution of $X + Y$.

If $X_1, \ldots, X_n$ are independent Poisson random variables with parameter $\lambda = 1$, find the distribution of $\sum_{i=1}^{n} X_i$. Hence or otherwise, find the limit of the real sequence

$$a_n = e^{-n} \sum_{j=0}^{n} \frac{n^j}{j!}, \quad n \in \mathbb{N}.$$

[Standard results may be used without proof provided they are clearly stated.]

11F Probability

For any function $g \colon \mathbb{R} \to \mathbb{R}$ and random variables X, Y, the "tower property" of conditional expectations is

$$E[g(X)] = E\big[E[g(X)|Y]\big].$$

Provide a proof of this property when both X, Y are discrete.

Let $U_1, U_2, \ldots$ be a sequence of independent uniform $U(0,1)$-random variables. For $x \in [0,1]$ find the expected number of U_i's needed such that their sum exceeds x, that is, find $E[N(x)]$ where

$$N(x) = \min\left\{ n : \sum_{i=1}^{n} U_i > x \right\}.$$

[*Hint: Write* $E[N(x)] = E\big[E[N(x)|U_1]\big]$.]

12F Probability

Give the definition of an exponential random variable X with parameter λ. Show that X is memoryless.

Now let X, Y be independent exponential random variables, each with parameter λ. Find the probability density function of the random variable $Z = \min(X, Y)$ and the probability $P(X > Y)$.

Suppose the random variables G_1, G_2 are independent and each has probability density function given by

$$f(y) = C^{-1}e^{-y}y^{-1/2}, \quad y > 0, \quad \text{where } C = \int_0^\infty e^{-y}y^{-1/2}dy.$$

Find the probability density function of $G_1 + G_2$. [You may use standard results without proof provided they are clearly stated.]

END OF PAPER