UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 29 May, 2014 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Vectors and Matrices

(a) Let

$$z = 2 + 2i.$$

(i) Compute z^4.

(ii) Find all complex numbers w such that $w^4 = z$.

(b) Find all the solutions of the equation

$$e^{2z^2} - 1 = 0.$$

(c) Let $z = x + iy$, $\bar{z} = x - iy$, $x, y \in \mathbb{R}$. Show that the equation of any line, and of any circle, may be written respectively as

$$Bz + \bar{B}\bar{z} + C = 0 \qquad \text{and} \qquad z\bar{z} + \bar{B}z + B\bar{z} + C = 0,$$

for some complex B and real C.

2A Vectors and Matrices

(a) What is meant by an eigenvector and the corresponding eigenvalue of a matrix A?

(b) Let A be the matrix

$$A = \begin{pmatrix} 3 & -2 & -2 \\ 1 & 0 & -2 \\ 3 & -3 & -1 \end{pmatrix}.$$

Find the eigenvalues and the corresponding eigenspaces of A and determine whether or not A is diagonalisable.

3D Analysis I

Show that every sequence of real numbers contains a monotone subsequence.

4F Analysis I

Find the radius of convergence of the following power series:

$$\text{(i)} \sum_{n \geqslant 1} \frac{n!}{n^n} z^n; \qquad \text{(ii)} \sum_{n \geqslant 1} n^n z^{n!}.$$

SECTION II

5B Vectors and Matrices

(i) For vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Show that the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ and the line $(\mathbf{r} - \mathbf{b}) \times \mathbf{m} = \mathbf{0}$, where $\mathbf{m} \cdot \mathbf{n} \neq 0$, intersect at the point

$$\mathbf{r} = \frac{(\mathbf{a} \cdot \mathbf{n})\mathbf{m} + \mathbf{n} \times (\mathbf{b} \times \mathbf{m})}{\mathbf{m} \cdot \mathbf{n}},$$

and only at that point. What happens if $\mathbf{m} \cdot \mathbf{n} = 0$?

(ii) Explain why the distance between the planes $(\mathbf{r} - \mathbf{a}_1) \cdot \hat{\mathbf{n}} = 0$ and $(\mathbf{r} - \mathbf{a}_2) \cdot \hat{\mathbf{n}} = 0$ is $\left|(\mathbf{a}_1 - \mathbf{a}_2) \cdot \hat{\mathbf{n}}\right|$, where $\hat{\mathbf{n}}$ is a unit vector.

(iii) Find the shortest distance between the lines $(3 + s, 3s, 4 - s)$ and $(-2, 3 + t, 3 - t)$ where $s, t \in \mathbb{R}$. [*You may wish to consider two appropriately chosen planes and use the result of part (ii).*]

6A Vectors and Matrices

Let A be a real $n \times n$ symmetric matrix.

(i) Show that all eigenvalues of A are real, and that the eigenvectors of A with respect to different eigenvalues are orthogonal. Assuming that any real symmetric matrix can be diagonalised, show that there exists an orthonormal basis $\{\mathbf{y}_i\}$ of eigenvectors of A.

(ii) Consider the linear system

$$A\mathbf{x} = \mathbf{b}.$$

Show that this system has a solution if and only if $\mathbf{b} \cdot \mathbf{h} = 0$ for every vector $\mathbf{h}$ in the kernel of A. Let $\mathbf{x}$ be such a solution. Given an eigenvector of A with non-zero eigenvalue, determine the component of $\mathbf{x}$ in the direction of this eigenvector. Use this result to find the general solution of the linear system, in the form

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{y}_i.$$

7C Vectors and Matrices

Let $\mathcal{A}\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map

$$\mathcal{A}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z\mathrm{e}^{i\theta} + w \\ w\mathrm{e}^{-i\phi} + z \end{pmatrix},$$

where θ and ϕ are real constants. Write down the matrix A of $\mathcal{A}$ with respect to the standard basis of $\mathbb{C}^2$ and show that $\det A = 2i \sin \frac{1}{2}(\theta - \phi) \exp\big(\frac{1}{2}i(\theta - \phi)\big)$.

Let $\mathcal{R}\colon \mathbb{C}^2 \to \mathbb{R}^4$ be the invertible map

$$\mathcal{R}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \operatorname{Re} z \\ \operatorname{Im} z \\ \operatorname{Re} w \\ \operatorname{Im} w \end{pmatrix}$$

and define a linear map $\mathcal{B}\colon \mathbb{R}^4 \to \mathbb{R}^4$ by $\mathcal{B} = \mathcal{R}\mathcal{A}\mathcal{R}^{-1}$. Find the image of each of the standard basis vectors of $\mathbb{R}^4$ under both $\mathcal{R}^{-1}$ and $\mathcal{B}$. Hence, or otherwise, find the matrix B of $\mathcal{B}$ with respect to the standard basis of $\mathbb{R}^4$ and verify that $\det B = |\det A|^2$.

8C Vectors and Matrices

Let A and B be complex $n \times n$ matrices.

(i) The *commutator* of A and B is defined to be

$$[A, B] \equiv AB - BA.$$

Show that $[A, A] = 0$; $[A, B] = -[B, A]$; and $[A, \lambda B] = \lambda[A, B]$ for $\lambda \in \mathbb{C}$. Show further that the trace of $[A, B]$ vanishes.

(ii) A *skew-Hermitian* matrix S is one which satisfies $S^\dagger = -S$, where $\dagger$ denotes the Hermitian conjugate. Show that if A and B are skew-Hermitian then so is $[A, B]$.

(iii) Let $\mathcal{M}$ be the linear map from $\mathbb{R}^3$ to the set of 2×2 complex matrices given by

$$\mathcal{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xM_1 + yM_2 + zM_3,$$

where

$$M_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad M_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Prove that for any $\mathbf{a} \in \mathbb{R}^3$, $\mathcal{M}(\mathbf{a})$ is traceless and skew-Hermitian. By considering pairs such as $[M_1, M_2]$, or otherwise, show that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,

$$\mathcal{M}(\mathbf{a} \times \mathbf{b}) = \big[\mathcal{M}(\mathbf{a}), \mathcal{M}(\mathbf{b})\big].$$

(iv) Using the result of part (iii), or otherwise, prove that if C is a traceless skew-Hermitian 2×2 matrix then there exist matrices A, B such that $C = [A, B]$. [You may use geometrical properties of vectors in $\mathbb{R}^3$ without proof.]

9D Analysis I

(a) Show that for all $x \in \mathbb{R}$,

$$\lim_{k \to \infty} 3^k \sin(x/3^k) = x,$$

stating carefully what properties of sin you are using.

Show that the series $\sum_{n \geqslant 1} 2^n \sin(x/3^n)$ converges absolutely for all $x \in \mathbb{R}$.

(b) Let $(a_n)_{n \in \mathbb{N}}$ be a decreasing sequence of positive real numbers tending to zero. Show that for $\theta \in \mathbb{R}$, θ not a multiple of 2π, the series

$$\sum_{n \geqslant 1} a_n e^{in\theta}$$

converges.

Hence, or otherwise, show that $\sum_{n \geqslant 1} \frac{\sin(n\theta)}{n}$ converges for all $\theta \in \mathbb{R}$.

10E Analysis I

(i) State the Mean Value Theorem. Use it to show that if $f\colon (a,b) \to \mathbb{R}$ is a differentiable function whose derivative is identically zero, then f is constant.

(ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and $\alpha > 0$ a real number such that for all x, $y \in \mathbb{R}$,

$$|f(x) - f(y)| \leqslant |x - y|^{\alpha}\,.$$

Show that f is continuous. Show moreover that if $\alpha > 1$ then f is constant.

(iii) Let $f\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). Assume also that the right derivative of f at a exists; that is, the limit

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$$

exists. Show that for any $\epsilon > 0$ there exists $x \in (a,b)$ satisfying

$$\left| \frac{f(x) - f(a)}{x - a} - f'(x) \right| < \epsilon.$$

[You should not assume that f' is continuous.]

11E Analysis I

(i) Prove Taylor's Theorem for a function $f\colon \mathbb{R} \to \mathbb{R}$ differentiable n times, in the following form: for every $x \in \mathbb{R}$ there exists θ with $0 < \theta < 1$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n)}(\theta x)}{n!} x^n.$$

[You may assume Rolle's Theorem and the Mean Value Theorem; other results should be proved.]

(ii) The function $f\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable, and satisfies the differential equation $f'' - f = 0$ with $f(0) = A$, $f'(0) = B$. Show that f is infinitely differentiable. Write down its Taylor series at the origin, and prove that it converges to f at every point. Hence or otherwise show that for any a, $h \in \mathbb{R}$, the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} h^k$$

converges to $f(a + h)$.

12F Analysis I

Define what it means for a function $f\colon [0,1] \to \mathbb{R}$ to be (Riemann) integrable. Prove that f is integrable whenever it is

(a) continuous,

(b) monotonic.

Let $\{q_k : k \in \mathbb{N}\}$ be an enumeration of all rational numbers in $[0,1)$. Define a function $f\colon [0,1] \to \mathbb{R}$ by $f(0) = 0$,

$$f(x) = \sum_{k \in Q(x)} 2^{-k}, \quad x \in (0,1],$$

where

$$Q(x) = \{k \in \mathbb{N} : q_k \in [0,x)\}.$$

Show that f has a point of discontinuity in every interval $I \subset [0,1]$.

Is f integrable? [Justify your answer.]

END OF PAPER