38C Waves

The function $\phi(x,t)$ satisfies the equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^4 \phi}{\partial x^2 \partial t^2} \,.$$

Derive the dispersion relation, and sketch graphs of frequency, phase velocity and group velocity as functions of the wavenumber. In the case of a localised initial disturbance, will it be the shortest or the longest waves that are to be found at the front of a dispersing wave packet? Do the wave crests move faster or slower than the wave packet?

Give the solution to the initial-value problem for which at $t = 0$

$$\phi = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk \quad \text{and} \quad \frac{\partial \phi}{\partial t} = 0 \,,$$

and $\phi(x,0)$ is real. Use the method of stationary phase to obtain an approximation for $\phi(Vt,t)$ for fixed $0 < V < 1$ and large t. If, in addition, $\phi(x,0) = \phi(-x,0)$, deduce an approximation for the sequence of times at which $\phi(Vt,t) = 0$.

You are given that $\phi(t,t)$ decreases like $t^{-1/3}$ for large t. Give a brief physical explanation why this rate of decay is slower than for $0 < V < 1$. What can be said about $\phi(Vt,t)$ for large t if $V > 1$? [Detailed calculation is not required in these cases.]

[*You may assume that* $\int_{-\infty}^{\infty} e^{-au^2} du = \sqrt{\frac{\pi}{a}}$ *for* $\mathrm{Re}(a) \geqslant 0,\ a \neq 0$.]