31D Partial Differential Equations

In this question, functions are all real-valued, and

$$H^s_{per} = \{u = \sum_{m\in\mathbb{Z}} \hat{u}(m)e^{imx} \in L^2 : \|u\|^2_{H^s} = \sum_{m\in\mathbb{Z}} (1+m^2)^s |\hat{u}(m)|^2 < \infty\}$$

are the Sobolev spaces of functions 2π-periodic in x, for $s = 0, 1, 2, \ldots$.

State Parseval's theorem. For $s = 0, 1$ prove that the norm $\|u\|_{H^s}$ is equivalent to the norm $\| \ \|_s$ defined by

$$\|u\|^2_s = \sum_{r=0}^{s} \int_{-\pi}^{+\pi} (\partial_x^r u)^2 \, dx \, .$$

Consider the Cauchy problem

$$u_t - u_{xx} = f \, , \qquad u(x,0) = u_0(x) \, , \qquad t \geqslant 0 \, , \tag{1}$$

where $f = f(x,t)$ is a smooth function which is 2π-periodic in x, and the initial value u_0 is also smooth and 2π-periodic. Prove that if u is a smooth solution which is 2π-periodic in x, then it satisfies

$$\int_0^T \int_{-\pi}^{\pi} \left(u_t^2 + u_{xx}^2 \right) dx \, dt \leqslant C \left(\|u_0\|^2_{H^1} + \int_0^T \int_{-\pi}^{\pi} |f(x,t)|^2 \, dx \, dt \right)$$

for some number $C > 0$ which does not depend on u or f.

State the Lax–Milgram lemma. Prove, using the Lax–Milgram lemma, that if

$$f(x,t) = e^{\lambda t} g(x)$$

with $g \in H^0_{per}$ and $\lambda > 0$, then there exists a weak solution to (1) of the form $u(x,t) = e^{\lambda t}\phi(x)$ with $\phi \in H^1_{per}$. Does the same hold for all $\lambda \in \mathbb{R}$? Briefly explain your answer.