UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2014

List of Courses

Analysis I

Differential Equations

Dynamics and Relativity

Groups

Numbers and Sets

Probability

Vector Calculus

Vectors and Matrices

Paper 1, Section I

3D Analysis I

Show that every sequence of real numbers contains a monotone subsequence.

Paper 1, Section I

4F Analysis I

Find the radius of convergence of the following power series:

$$\text{(i)}\ \sum_{n\geqslant 1} \frac{n!}{n^n} z^n; \qquad \text{(ii)}\ \sum_{n\geqslant 1} n^n z^{n!}.$$

Paper 1, Section II

9D Analysis I

(a) Show that for all $x \in \mathbb{R}$,

$$\lim_{k\to\infty} 3^k \sin(x/3^k) = x,$$

stating carefully what properties of sin you are using.

Show that the series $\sum_{n\geqslant 1} 2^n \sin(x/3^n)$ converges absolutely for all $x \in \mathbb{R}$.

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers tending to zero. Show that for $\theta \in \mathbb{R}$, θ not a multiple of 2π, the series

$$\sum_{n\geqslant 1} a_n e^{in\theta}$$

converges.

Hence, or otherwise, show that $\sum_{n\geqslant 1} \frac{\sin(n\theta)}{n}$ converges for all $\theta \in \mathbb{R}$.

Paper 1, Section II

10E Analysis I

(i) State the Mean Value Theorem. Use it to show that if $f\colon (a,b) \to \mathbb{R}$ is a differentiable function whose derivative is identically zero, then f is constant.

(ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and $\alpha > 0$ a real number such that for all x, $y \in \mathbb{R}$,

$$|f(x) - f(y)| \leqslant |x - y|^{\alpha}.$$

Show that f is continuous. Show moreover that if $\alpha > 1$ then f is constant.

(iii) Let $f\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). Assume also that the right derivative of f at a exists; that is, the limit

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$$

exists. Show that for any $\epsilon > 0$ there exists $x \in (a,b)$ satisfying

$$\left| \frac{f(x) - f(a)}{x - a} - f'(x) \right| < \epsilon.$$

[You should not assume that f' is continuous.]

Paper 1, Section II

11E Analysis I

(i) Prove Taylor's Theorem for a function $f\colon \mathbb{R} \to \mathbb{R}$ differentiable n times, in the following form: for every $x \in \mathbb{R}$ there exists θ with $0 < \theta < 1$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n)}(\theta x)}{n!} x^n.$$

[You may assume Rolle's Theorem and the Mean Value Theorem; other results should be proved.]

(ii) The function $f\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable, and satisfies the differential equation $f'' - f = 0$ with $f(0) = A$, $f'(0) = B$. Show that f is infinitely differentiable. Write down its Taylor series at the origin, and prove that it converges to f at every point. Hence or otherwise show that for any a, $h \in \mathbb{R}$, the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} h^k$$

converges to $f(a+h)$.

Paper 1, Section II

12F Analysis I

Define what it means for a function $f\colon [0,1] \to \mathbb{R}$ to be (Riemann) integrable. Prove that f is integrable whenever it is

(a) continuous,

(b) monotonic.

Let $\{q_k : k \in \mathbb{N}\}$ be an enumeration of all rational numbers in $[0,1)$. Define a function $f\colon [0,1] \to \mathbb{R}$ by $f(0) = 0$,

$$f(x) = \sum_{k \in Q(x)} 2^{-k}, \quad x \in (0,1],$$

where

$$Q(x) = \{k \in \mathbb{N} : q_k \in [0,x)\}.$$

Show that f has a point of discontinuity in every interval $I \subset [0,1]$.

Is f integrable? [Justify your answer.]

Paper 2, Section I

1B Differential Equations

The following equation arises in the theory of elastic beams:

$$t^4 \frac{d^2u}{dt^2} + \lambda^2 u = 0, \quad \lambda > 0,\ t > 0,$$

where $u(t)$ is a real valued function.

By using the change of variables

$$t = \frac{1}{\tau}, \quad u(t) = \frac{v(\tau)}{\tau},$$

find the general solution of the above equation.

Paper 2, Section I

2B Differential Equations

Consider the ordinary differential equation

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0. \qquad (*)$$

State an equation to be satisfied by P and Q that ensures that equation $(*)$ is exact. In this case, express the general solution of equation $(*)$ in terms of a function $F(x,y)$ which should be defined in terms of P and Q.

Consider the equation

$$\frac{dy}{dx} = -\frac{4x+3y}{3x+3y^2},$$

satisfying the boundary condition $y(1) = 2$. Find an explicit relation between y and x.

Paper 2, Section II

5B Differential Equations

Use the transformation

$$y(t) = \frac{1}{cx(t)} \frac{dx(t)}{dt},$$

where c is a constant, to map the Ricatti equation

$$\frac{dy}{dt} + cy^2 + a(t)y + b(t) = 0, \quad t > 0,$$

to a linear equation.

Using the above result, as well as the change of variables $\tau = \ln t$, solve the boundary value problem

$$\frac{dy}{dt} + y^2 + \frac{y}{t} - \frac{\lambda^2}{t^2} = 0, \quad t > 0,$$

$$y(1) = 2\lambda,$$

where λ is a positive constant. What is the value of $t > 0$ for which the solution is singular?

Paper 2, Section II

6B Differential Equations

The so-called "shallow water theory" is characterised by the equations

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left[(h+\zeta)u\right] = 0,$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \zeta}{\partial x} = 0,$$

where g denotes the gravitational constant, the constant h denotes the undisturbed depth of the water, $u(x,t)$ denotes the speed in the x-direction, and $\zeta(x,t)$ denotes the elevation of the water.

(i) Assuming that $|u|$ and $|\zeta|$ and their gradients are small in some appropriate dimensional considerations, show that ζ satisfies the wave equation

$$\frac{\partial^2 \zeta}{\partial t^2} = c^2\,\frac{\partial^2 \zeta}{\partial x^2}, \qquad (*)$$

where the constant c should be determined in terms of h and g.

(ii) Using the change of variables

$$\xi = x + ct, \quad \eta = x - ct,$$

show that the general solution of $(*)$ satisfying the initial conditions

$$\zeta(x,0) = u_0(x), \quad \frac{\partial \zeta}{\partial t}(x,0) = v_0(x),$$

is given by

$$\zeta(x,t) = f(x+ct) + g(x-ct),$$

where

$$\frac{df(x)}{dx} = \frac{1}{2}\left[\frac{du_0(x)}{dx} + \frac{1}{c}v_0(x)\right],$$
$$\frac{dg(x)}{dx} = \frac{1}{2}\left[\frac{du_0(x)}{dx} - \frac{1}{c}v_0(x)\right].$$

Simplify the above to find ζ in terms of u_0 and v_0.

(iii) Find $\zeta(x,t)$ in the particular case that

$$u_0(x) = H(x+1) - H(x-1), \quad v_0(x) = 0, \quad -\infty < x < \infty,$$

where $H(\cdot)$ denotes the Heaviside step function.

Describe in words this solution.

Paper 2, Section II

7B Differential Equations

(a) Let $y_1(x)$ be a solution of the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

Assuming that the second linearly independent solution takes the form $y_2(x) = v(x)y_1(x)$, derive an ordinary differential equation for $v(x)$.

(b) Consider the equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \quad -1 < x < 1.$$

By inspection or otherwise, find an explicit solution of this equation. Use the result in (a) to find the solution $y(x)$ satisfying the conditions

$$y(0) = \frac{dy}{dx}(0) = 1.$$

Paper 2, Section II

8B Differential Equations

Consider the damped pendulum equation

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \sin\theta = 0, \qquad (*)$$

where c is a positive constant. The energy E, which is the sum of the kinetic energy and the potential energy, is defined by

$$E(t) = \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + 1 - \cos\theta.$$

(i) Verify that $E(t)$ is a decreasing function.

(ii) Assuming that θ is sufficiently small, so that terms of order θ^3 can be neglected, find an approximation for the general solution of $(*)$ in terms of two arbitrary constants. Discuss the dependence of this approximate solution on c.

(iii) By rewriting $(*)$ as a system of equations for $x(t) = \theta$ and $y(t) = \dot{\theta}$, find all stationary points of $(*)$ and discuss their nature for all c, except $c = 2$.

(iv) Draw the phase plane curves for the particular case $c = 1$.

Paper 4, Section I

3C Dynamics and Relativity

A particle of mass m has charge q and moves in a constant magnetic field $\mathbf{B}$. Show that the particle's path describes a helix. In which direction is the axis of the helix, and what is the particle's rotational angular frequency about that axis?

Paper 4, Section I

4C Dynamics and Relativity

What is a *4-vector*? Define the inner product of two 4-vectors and give the meanings of the terms *timelike*, *null* and *spacelike*. How do the four components of a 4-vector change under a Lorentz transformation of speed v? [Without loss of generality, you may take the velocity of the transformation to be along the positive x-axis.]

Show that a 4-vector that is timelike in one frame of reference is also timelike in a second frame of reference related by a Lorentz transformation. [Again, you may without loss of generality take the velocity of the transformation to be along the positive x-axis.]

Show that any null 4-vector may be written in the form $a(1, \hat{\mathbf{n}})$ where a is real and $\hat{\mathbf{n}}$ is a unit 3-vector. Given any two null 4-vectors that are *future-pointing*, that is, which have positive time-components, show that their sum is either null or timelike.

Paper 4, Section II

9C Dynamics and Relativity

A rocket of mass $m(t)$, which includes the mass of its fuel and everything on board, moves through free space in a straight line at speed $v(t)$. When its engines are operational, they burn fuel at a constant mass rate α and eject the waste gases behind the rocket at a constant speed u relative to the rocket. Obtain the rocket equation

$$m\frac{dv}{dt} - \alpha u = 0.$$

The rocket is initially at rest in a cloud of space dust which is also at rest. The engines are started and, as the rocket travels through the cloud, it collects dust which it stores on board for research purposes. The mass of dust collected in a time δt is given by $\beta\,\delta x$, where δx is the distance travelled in that time and β is a constant. Obtain the new equations

$$\begin{aligned}\frac{dm}{dt} &= \beta v - \alpha,\\ m\frac{dv}{dt} &= \alpha u - \beta v^2.\end{aligned}$$

By eliminating t, or otherwise, obtain the relationship

$$m = \lambda m_0 u \sqrt{\frac{(\lambda u - v)^{\lambda - 1}}{(\lambda u + v)^{\lambda + 1}}}\,,$$

where m_0 is the initial mass of the rocket and $\lambda = \sqrt{\alpha/\beta u}$.

If $\lambda > 1$, show that the fuel will be exhausted before the speed of the rocket can reach λu. Comment on the case when $\lambda < 1$, giving a physical interpretation of your answer.

Paper 4, Section II

10C Dynamics and Relativity

A reference frame S' rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame S that has the same origin as S'. A particle of mass m at position vector $\mathbf{x}$ is subject to a force $\mathbf{F}$. Derive the equation of motion for the particle in S'.

A marble moves on a smooth plane which is inclined at an angle θ to the horizontal. The whole plane rotates at constant angular speed ω about a vertical axis through a point O fixed in the plane. Coordinates (ξ, η) are defined with respect to axes fixed in the plane: $O\xi$ horizontal and $O\eta$ up the line of greatest slope in the plane. Ensuring that you account for the normal reaction force, show that the motion of the marble obeys

$$\begin{aligned}\ddot{\xi} &= \omega^2 \xi + 2\omega\dot{\eta}\cos\theta,\\ \ddot{\eta} &= \omega^2 \eta \cos^2\theta - 2\omega\dot{\xi}\cos\theta - g\sin\theta.\end{aligned}$$

By considering the marble's kinetic energy as measured on the plane in the rotating frame, or otherwise, find a constant of the motion.

[You may assume that the marble never leaves the plane.]

Paper 4, Section II

11C Dynamics and Relativity

A thin flat disc of radius a has density (mass per unit area) $\rho(r,\theta) = \rho_0(a-r)$ where (r,θ) are plane polar coordinates on the disc and ρ_0 is a constant. The disc is free to rotate about a light, thin rod that is rigidly fixed in space, passing through the centre of the disc orthogonal to it. Find the moment of inertia of the disc about the rod.

The section of the disc lying in $r \geqslant \frac{1}{2}a$, $-\frac{\pi}{13} \leqslant \theta \leqslant \frac{\pi}{13}$ is cut out and removed. Starting from rest, a constant torque τ is applied to the remaining part of the disc until its angular speed about the axis reaches Ω. Show that this takes a time

$$\frac{3\pi\rho_0 a^5 \Omega}{32\tau}.$$

After this time, no further torque is applied and the partial disc continues to rotate at constant angular speed Ω. Given that the total mass of the partial disc is $k\rho_0 a^3$, where k is a constant that you need not determine, find the position of the centre of mass, and hence its acceleration. From where does the force required to produce this acceleration arise?

Paper 4, Section II

12C Dynamics and Relativity

Define the *4-momentum* of a particle and describe briefly the principle of conservation of 4-momentum.

A photon of angular frequency ω is absorbed by a particle of rest mass m that is stationary in the laboratory frame of reference. The particle then splits into two equal particles, each of rest mass αm.

Find the maximum possible value of α as a function of $\mu = \hbar\omega/mc^2$. Verify that as $\mu \to 0$, this maximum value tends to $\frac{1}{2}$. For general μ, show that when the maximum value of α is achieved, the resulting particles are each travelling at speed $c/(1+\mu^{-1})$ in the laboratory frame.

Paper 3, Section I

1D Groups

Let $G = \mathbb{Q}$ be the rational numbers, with addition as the group operation. Let x, y be non-zero elements of G, and let $N \leqslant G$ be the subgroup they generate. Show that N is isomorphic to $\mathbb{Z}$.

Find non-zero elements $x, y \in \mathbb{R}$ which generate a subgroup that is not isomorphic to $\mathbb{Z}$.

Paper 3, Section I

2D Groups

Let G be a group, and suppose the centre of G is trivial. If p divides $|G|$, show that G has a non-trivial conjugacy class whose order is prime to p.

Paper 3, Section II

5D Groups

Let S_n be the group of permutations of $\{1, \ldots, n\}$, and suppose n is even, $n \geqslant 4$.

Let $g = (1\,2) \in S_n$, and $h = (1\,2)(3\,4)\ldots(n{-}1\;\,n) \in S_n$.

(i) Compute the centraliser of g, and the orders of the centraliser of g and of the centraliser of h.

(ii) Now let $n = 6$. Let G be the group of all symmetries of the cube, and X the set of faces of the cube. Show that the action of G on X makes G isomorphic to the centraliser of h in S_6. [*Hint: Show that* $-1 \in G$ *permutes the faces of the cube according to* h.]

Show that G is also isomorphic to the centraliser of g in S_6.

Paper 3, Section II

6D Groups

Let p be a prime number. Let G be a group such that every non-identity element of G has order p.

(i) Show that if $|G|$ is finite, then $|G| = p^n$ for some n. [You must prove any theorems that you use.]

(ii) Show that if $H \leqslant G$, and $x \notin H$, then $\langle x \rangle \cap H = \{1\}$.

Hence show that if G is abelian, and $|G|$ is finite, then $G \simeq C_p \times \cdots \times C_p$.

(iii) Let G be the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b, x \in \mathbb{F}_p$ and $\mathbb{F}_p$ is the field of integers modulo p. Show that every non-identity element of G has order p if and only if $p > 2$. [You may assume that G is a subgroup of the group of all 3×3 invertible matrices.]

Paper 3, Section II

7D Groups

Let p be a prime number, and $G = GL_2(\mathbb{F}_p)$, the group of 2×2 invertible matrices with entries in the field $\mathbb{F}_p$ of integers modulo p.

The group G acts on $X = \mathbb{F}_p \cup \{\infty\}$ by Möbius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

(i) Show that given any distinct $x, y, z \in X$ there exists $g \in G$ such that $g \cdot 0 = x$, $g \cdot 1 = y$ and $g \cdot \infty = z$. How many such g are there?

(ii) G acts on $X \times X \times X$ by $g \cdot (x, y, z) = (g \cdot x, g \cdot y, g \cdot z)$. Describe the orbits, and for each orbit, determine its stabiliser, and the orders of the orbit and stabiliser.

Paper 3, Section II

8D Groups

(a) Let G be a group, and N a subgroup of G. Define what it means for N to be normal in G, and show that if N is normal then G/N naturally has the structure of a group.

(b) For each of (i)–(iii) below, give an example of a non-trivial finite group G and non-trivial normal subgroup $N \leqslant G$ satisfying the stated properties.

(i) $G/N \times N \simeq G$.

(ii) There is no group homomorphism $G/N \to G$ such that the composite $G/N \to G \to G/N$ is the identity.

(iii) There is a group homomorphism $i\colon G/N \to G$ such that the composite $G/N \to G \to G/N$ is the identity, but the map

$$G/N \times N \to G, \qquad (gN, n) \mapsto i(gN)n$$

is not a group homomorphism.

Show also that for any $N \leqslant G$ satisfying (iii), this map is always a bijection.

Paper 4, Section I

1E Numbers and Sets

Use Euclid's algorithm to determine d, the greatest common divisor of 203 and 147, and to express it in the form $203x + 147y$ for integers x, y. Hence find all solutions in integers x, y of the equation $203x + 147y = d$.

How many integers n are there with $1 \leqslant n \leqslant 2014$ and $21n \equiv 25 \pmod{29}$?

Paper 4, Section I

2E Numbers and Sets

Define the binomial coefficients $\binom{n}{k}$, for integers n, k satisfying $n \geqslant k \geqslant 0$. Prove directly from your definition that if $n > k \geqslant 0$ then

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

and that for every $m \geqslant 0$ and $n \geqslant 0$,

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}.$$

Paper 4, Section II

5E Numbers and Sets

What does it mean to say that the sequence of real numbers (x_n) converges to the limit x? What does it mean to say that the series $\sum_{n=1}^{\infty} x_n$ converges to s?

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series of positive real numbers. Suppose that (x_n) is a sequence of positive real numbers such that for every $n \geqslant 1$, either $x_n \leqslant a_n$ or $x_n \leqslant b_n$. Show that $\sum_{n=1}^{\infty} x_n$ is convergent.

Show that $\sum_{n=1}^{\infty} 1/n^2$ is convergent, and that $\sum_{n=1}^{\infty} 1/n^{\alpha}$ is divergent if $\alpha \leqslant 1$.

Let (x_n) be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} n^2 x_n^2$ is convergent. Show that $\sum_{n=1}^{\infty} x_n$ is convergent. Determine (with proof or counterexample) whether or not the converse statement holds.

Paper 4, Section II

6E Numbers and Sets

(i) State and prove the Fermat–Euler Theorem.

(ii) Let p be an odd prime number, and x an integer coprime to p. Show that $x^{(p-1)/2} \equiv \pm 1 \pmod p$, and that if the congruence $y^2 \equiv x \pmod p$ has a solution then $x^{(p-1)/2} \equiv 1 \pmod p$.

(iii) By arranging the residue classes coprime to p into pairs $\{a, bx\}$ with $ab \equiv 1 \pmod p$, or otherwise, show that if the congruence $y^2 \equiv x \pmod p$ has no solution then $x^{(p-1)/2} \equiv -1 \pmod p$.

(iv) Show that $5^{5^5} \equiv 5 \pmod{23}$.

Paper 4, Section II

7E Numbers and Sets

(i) What does it mean to say that a set X is countable? Show directly that the set of sequences $(x_n)_{n\in\mathbb{N}}$, with $x_n \in \{0,1\}$ for all n, is uncountable.

(ii) Let S be any subset of $\mathbb{N}$. Show that there exists a bijection $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(S) = 2\mathbb{N}$ (the set of even natural numbers) if and only if both S and its complement are infinite.

(iii) Let $\sqrt{2} = 1{\cdot}a_1a_2a_3\ldots$ be the binary expansion of $\sqrt{2}$. Let X be the set of all sequences (x_n) with $x_n \in \{0,1\}$ such that for infinitely many n, $x_n = 0$. Let Y be the set of all $(x_n) \in X$ such that for infinitely many n, $x_n = a_n$. Show that Y is uncountable.

Paper 4, Section II

8E Numbers and Sets

(i) State and prove the Inclusion–Exclusion Principle.

(ii) Let $n > 1$ be an integer. Denote by $\mathbb{Z}/n\mathbb{Z}$ the integers modulo n. Let X be the set of all functions $f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ such that for every $j \in \mathbb{Z}/n\mathbb{Z}$, $f(j) - f(j-1) \not\equiv j \pmod n$. Show that

$$|X| = \begin{cases} (n-1)^n + 1 - n & \text{if } n \text{ is odd,} \\ (n-1)^n - 1 & \text{if } n \text{ is even.} \end{cases}$$

Paper 2, Section I

3F Probability

Consider a particle situated at the origin $(0,0)$ of $\mathbb{R}^2$. At successive times a direction is chosen independently by picking an angle uniformly at random in the interval $[0,2\pi]$, and the particle then moves an Euclidean unit length in this direction. Find the expected squared Euclidean distance of the particle from the origin after n such movements.

Paper 2, Section I

4F Probability

Consider independent discrete random variables $X_1,\ldots,X_n$ and assume $E[X_i]$ exists for all $i=1,\ldots,n$.

Show that

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i].$$

If the $X_1,\ldots,X_n$ are also positive, show that

$$\prod_{i=1}^{n}\sum_{m=0}^{\infty} P(X_i > m) = \sum_{m=0}^{\infty} P\left(\prod_{i=1}^{n} X_i > m\right).$$

Paper 2, Section II

9F Probability

State the axioms of probability.

State and prove Boole's inequality.

Suppose you toss a sequence of coins, the i-th of which comes up heads with probability p_i, where $\sum_{i=1}^{\infty} p_i < \infty$. Calculate the probability of the event that infinitely many heads occur.

Suppose you repeatedly and independently roll a pair of fair dice and each time record the sum of the dice. What is the probability that an outcome of 5 appears before an outcome of 7? Justify your answer.

Paper 2, Section II

10F Probability

Define what it means for a random variable X to have a Poisson distribution, and find its moment generating function.

Suppose X, Y are independent Poisson random variables with parameters λ, μ. Find the distribution of $X + Y$.

If $X_1, \ldots, X_n$ are independent Poisson random variables with parameter $\lambda = 1$, find the distribution of $\sum_{i=1}^{n} X_i$. Hence or otherwise, find the limit of the real sequence

$$a_n = e^{-n} \sum_{j=0}^{n} \frac{n^j}{j!}, \quad n \in \mathbb{N}.$$

[Standard results may be used without proof provided they are clearly stated.]

Paper 2, Section II

11F Probability

For any function $g\colon \mathbb{R} \to \mathbb{R}$ and random variables X, Y, the "tower property" of conditional expectations is

$$E[g(X)] = E\big[E[g(X)|Y]\big].$$

Provide a proof of this property when both X, Y are discrete.

Let $U_1, U_2, \ldots$ be a sequence of independent uniform $U(0,1)$-random variables. For $x \in [0,1]$ find the expected number of U_i's needed such that their sum exceeds x, that is, find $E[N(x)]$ where

$$N(x) = \min\left\{ n : \sum_{i=1}^{n} U_i > x \right\}.$$

[*Hint: Write* $E[N(x)] = E\big[E[N(x)|U_1]\big]$.]

Paper 2, Section II

12F Probability

Give the definition of an exponential random variable X with parameter λ. Show that X is memoryless.

Now let X, Y be independent exponential random variables, each with parameter λ. Find the probability density function of the random variable $Z = \min(X, Y)$ and the probability $P(X > Y)$.

Suppose the random variables G_1, G_2 are independent and each has probability density function given by

$$f(y) = C^{-1} e^{-y} y^{-1/2}, \quad y > 0, \quad \text{where } C = \int_0^\infty e^{-y} y^{-1/2} dy.$$

Find the probability density function of $G_1 + G_2$. [You may use standard results without proof provided they are clearly stated.]

Paper 3, Section I

3A Vector Calculus

(a) For $\mathbf{x} \in \mathbb{R}^n$ and $r = |\mathbf{x}|$, show that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}.$$

(b) Use index notation and your result in (a), or otherwise, to compute

(i) $\nabla \cdot \big(f(r)\mathbf{x}\big)$, and

(ii) $\nabla \times \big(f(r)\mathbf{x}\big)$ for $n = 3$.

(c) Show that for each $n \in \mathbb{N}$ there is, up to an arbitrary constant, just one vector field $\mathbf{F}(\mathbf{x})$ of the form $f(r)\mathbf{x}$ such that $\nabla \cdot \mathbf{F}(\mathbf{x}) = 0$ everywhere on $\mathbb{R}^n \backslash \{\mathbf{0}\}$, and determine $\mathbf{F}$.

Paper 3, Section I

4A Vector Calculus

Let $\mathbf{F}(\mathbf{x})$ be a vector field defined everywhere on the domain $G \subset \mathbb{R}^3$.

(a) Suppose that $\mathbf{F}(\mathbf{x})$ has a potential $\phi(\mathbf{x})$ such that $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ for $\mathbf{x} \in G$. Show that

$$\int_\gamma \mathbf{F} \cdot \mathbf{dx} = \phi(\mathbf{b}) - \phi(\mathbf{a})$$

for any smooth path γ from $\mathbf{a}$ to $\mathbf{b}$ in G. Show further that necessarily $\nabla \times \mathbf{F} = \mathbf{0}$ on G.

(b) State a condition for G which ensures that $\nabla \times \mathbf{F} = \mathbf{0}$ implies $\int_\gamma \mathbf{F} \cdot \mathbf{dx}$ is path-independent.

(c) Compute the line integral $\oint_\gamma \mathbf{F} \cdot \mathbf{dx}$ for the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix},$$

where γ denotes the anti-clockwise path around the unit circle in the (x, y)-plane. Compute $\nabla \times \mathbf{F}$ and comment on your result in the light of (b).

Paper 3, Section II

9A Vector Calculus

The surface C in $\mathbb{R}^3$ is given by $z^2 = x^2 + y^2$.

(a) Show that the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is tangent to the surface C everywhere.

(b) Show that the surface integral $\int_S \mathbf{F} \cdot \mathbf{dS}$ is a constant independent of S for any surface S which is a subset of C, and determine this constant.

(c) The volume V in $\mathbb{R}^3$ is bounded by the surface C and by the cylinder $x^2 + y^2 = 1$. Sketch V and compute the volume integral

$$\int_V \nabla \cdot \mathbf{F} \, dV$$

directly by integrating over V.

(d) Use the Divergence Theorem to verify the result you obtained in part (b) for the integral $\int_S \mathbf{F} \cdot \mathbf{dS}$, where S is the portion of C lying in $-1 \leqslant z \leqslant 1$.

Paper 3, Section II

10A Vector Calculus

(a) State Stokes' Theorem for a surface S with boundary ∂S.

(b) Let S be the surface in $\mathbb{R}^3$ given by $z^2 = 1 + x^2 + y^2$ where $\sqrt{2} \leqslant z \leqslant \sqrt{5}$. Sketch the surface S and find the surface element $\mathbf{dS}$ with respect to the Cartesian coordinates x and y.

(c) Compute $\nabla \times \mathbf{F}$ for the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} -y \\ x \\ xy(x+y) \end{pmatrix}$$

and verify Stokes' Theorem for $\mathbf{F}$ on the surface S.

Paper 3, Section II

11A Vector Calculus

(i) Starting with Poisson's equation in $\mathbb{R}^3$,

$$\nabla^2\phi(\mathbf{x}) = f(\mathbf{x}),$$

derive Gauss' flux theorem

$$\int_V f(\mathbf{x})\, dV = \int_{\partial V} \mathbf{F}(\mathbf{x}) \cdot \mathbf{dS}$$

for $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ and for any volume $V \subseteq \mathbb{R}^3$.

(ii) Let

$$I = \int_S \frac{\mathbf{x}\cdot\mathbf{dS}}{|\mathbf{x}|^3}.$$

Show that $I = 4\pi$ if S is the sphere $|\mathbf{x}| = R$, and that $I = 0$ if S bounds a volume that does not contain the origin.

(iii) Show that the electric field defined by

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^3}, \quad \mathbf{x} \neq \mathbf{a},$$

satisfies

$$\int_{\partial V} \mathbf{E}\cdot\mathbf{dS} = \begin{cases} 0 & \text{if } \mathbf{a} \notin V \\ \dfrac{q}{\epsilon_0} & \text{if } \mathbf{a} \in V \end{cases}$$

where ∂V is a surface bounding a closed volume V and $\mathbf{a} \notin \partial V$, and where the electric charge q and permittivity of free space ϵ_0 are constants. This is Gauss' law for a point electric charge.

(iv) Assume that $f(\mathbf{x})$ is spherically symmetric around the origin, i.e., it is a function only of $|\mathbf{x}|$. Assume that $\mathbf{F}(\mathbf{x})$ is also spherically symmetric. Show that $\mathbf{F}(\mathbf{x})$ depends only on the values of f inside the sphere with radius $|\mathbf{x}|$ but not on the values of f outside this sphere.

Paper 3, Section II

12A Vector Calculus

(a) Show that any rank 2 tensor t_{ij} can be written uniquely as a sum of two rank 2 tensors s_{ij} and a_{ij} where s_{ij} is symmetric and a_{ij} is antisymmetric.

(b) Assume that the rank 2 tensor t_{ij} is invariant under any rotation about the z-axis, as well as under a rotation of angle π about any axis in the (x, y)-plane through the origin.

(i) Show that there exist $\alpha, \beta \in \mathbb{R}$ such that t_{ij} can be written as

$$t_{ij} = \alpha\delta_{ij} + \beta\delta_{i3}\delta_{j3}. \qquad (*)$$

(ii) Is there some proper subgroup of the rotations specified above for which the result $(*)$ still holds if the invariance of t_{ij} is restricted to this subgroup? If so, specify the smallest such subgroup.

(c) The array of numbers d_{ijk} is such that $d_{ijk}s_{ij}$ is a vector for any symmetric matrix s_{ij}.

(i) By writing d_{ijk} as a sum of d^s_{ijk} and d^a_{ijk} with $d^s_{ijk} = d^s_{jik}$ and $d^a_{ijk} = -d^a_{jik}$, show that d^s_{ijk} is a rank 3 tensor. [You may assume without proof the Quotient Theorem for tensors.]

(ii) Does d^a_{ijk} necessarily have to be a tensor? Justify your answer.

Paper 1, Section I

1B Vectors and Matrices

(a) Let

$$z = 2 + 2i.$$

(i) Compute z^4.

(ii) Find all complex numbers w such that $w^4 = z$.

(b) Find all the solutions of the equation

$$e^{2z^2} - 1 = 0.$$

(c) Let $z = x + iy$, $\bar{z} = x - iy$, $x, y \in \mathbb{R}$. Show that the equation of any line, and of any circle, may be written respectively as

$$Bz + \bar{B}\bar{z} + C = 0 \qquad \text{and} \qquad z\bar{z} + \bar{B}z + B\bar{z} + C = 0,$$

for some complex B and real C.

Paper 1, Section I

2A Vectors and Matrices

(a) What is meant by an eigenvector and the corresponding eigenvalue of a matrix A?

(b) Let A be the matrix

$$A = \begin{pmatrix} 3 & -2 & -2 \\ 1 & 0 & -2 \\ 3 & -3 & -1 \end{pmatrix}.$$

Find the eigenvalues and the corresponding eigenspaces of A and determine whether or not A is diagonalisable.

Paper 1, Section II

5B Vectors and Matrices

(i) For vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Show that the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ and the line $(\mathbf{r} - \mathbf{b}) \times \mathbf{m} = \mathbf{0}$, where $\mathbf{m} \cdot \mathbf{n} \neq 0$, intersect at the point

$$\mathbf{r} = \frac{(\mathbf{a} \cdot \mathbf{n})\mathbf{m} + \mathbf{n} \times (\mathbf{b} \times \mathbf{m})}{\mathbf{m} \cdot \mathbf{n}},$$

and only at that point. What happens if $\mathbf{m} \cdot \mathbf{n} = 0$?

(ii) Explain why the distance between the planes $(\mathbf{r} - \mathbf{a}_1) \cdot \hat{\mathbf{n}} = 0$ and $(\mathbf{r} - \mathbf{a}_2) \cdot \hat{\mathbf{n}} = 0$ is $\left|(\mathbf{a}_1 - \mathbf{a}_2) \cdot \hat{\mathbf{n}}\right|$, where $\hat{\mathbf{n}}$ is a unit vector.

(iii) Find the shortest distance between the lines $(3 + s, 3s, 4 - s)$ and $(-2, 3 + t, 3 - t)$ where $s, t \in \mathbb{R}$. [*You may wish to consider two appropriately chosen planes and use the result of part (ii).*]

Paper 1, Section II

6A Vectors and Matrices

Let A be a real $n \times n$ symmetric matrix.

(i) Show that all eigenvalues of A are real, and that the eigenvectors of A with respect to different eigenvalues are orthogonal. Assuming that any real symmetric matrix can be diagonalised, show that there exists an orthonormal basis $\{\mathbf{y}_i\}$ of eigenvectors of A.

(ii) Consider the linear system

$$A\mathbf{x} = \mathbf{b}.$$

Show that this system has a solution if and only if $\mathbf{b} \cdot \mathbf{h} = 0$ for every vector $\mathbf{h}$ in the kernel of A. Let $\mathbf{x}$ be such a solution. Given an eigenvector of A with non-zero eigenvalue, determine the component of $\mathbf{x}$ in the direction of this eigenvector. Use this result to find the general solution of the linear system, in the form

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{y}_i.$$

Paper 1, Section II

7C Vectors and Matrices

Let $\mathcal{A}\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map

$$\mathcal{A}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z\mathrm{e}^{i\theta} + w \\ w\mathrm{e}^{-i\phi} + z \end{pmatrix},$$

where θ and ϕ are real constants. Write down the matrix A of $\mathcal{A}$ with respect to the standard basis of $\mathbb{C}^2$ and show that $\det A = 2i \sin \frac{1}{2}(\theta - \phi) \exp\big(\frac{1}{2} i(\theta - \phi)\big)$.

Let $\mathcal{R}\colon \mathbb{C}^2 \to \mathbb{R}^4$ be the invertible map

$$\mathcal{R}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \operatorname{Re} z \\ \operatorname{Im} z \\ \operatorname{Re} w \\ \operatorname{Im} w \end{pmatrix}$$

and define a linear map $\mathcal{B}\colon \mathbb{R}^4 \to \mathbb{R}^4$ by $\mathcal{B} = \mathcal{R}\mathcal{A}\mathcal{R}^{-1}$. Find the image of each of the standard basis vectors of $\mathbb{R}^4$ under both $\mathcal{R}^{-1}$ and $\mathcal{B}$. Hence, or otherwise, find the matrix B of $\mathcal{B}$ with respect to the standard basis of $\mathbb{R}^4$ and verify that $\det B = |\det A|^2$.

Paper 1, Section II

8C Vectors and Matrices

Let A and B be complex $n \times n$ matrices.

(i) The *commutator* of A and B is defined to be

$$[A, B] \equiv AB - BA.$$

Show that $[A, A] = 0$; $[A, B] = -[B, A]$; and $[A, \lambda B] = \lambda[A, B]$ for $\lambda \in \mathbb{C}$. Show further that the trace of $[A, B]$ vanishes.

(ii) A *skew-Hermitian* matrix S is one which satisfies $S^\dagger = -S$, where $\dagger$ denotes the Hermitian conjugate. Show that if A and B are skew-Hermitian then so is $[A, B]$.

(iii) Let $\mathcal{M}$ be the linear map from $\mathbb{R}^3$ to the set of 2×2 complex matrices given by

$$\mathcal{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xM_1 + yM_2 + zM_3,$$

where

$$M_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad M_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Prove that for any $\mathbf{a} \in \mathbb{R}^3$, $\mathcal{M}(\mathbf{a})$ is traceless and skew-Hermitian. By considering pairs such as $[M_1, M_2]$, or otherwise, show that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,

$$\mathcal{M}(\mathbf{a} \times \mathbf{b}) = \big[\mathcal{M}(\mathbf{a}), \mathcal{M}(\mathbf{b})\big].$$

(iv) Using the result of part (iii), or otherwise, prove that if C is a traceless skew-Hermitian 2×2 matrix then there exist matrices A, B such that $C = [A, B]$. [You may use geometrical properties of vectors in $\mathbb{R}^3$ without proof.]