UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday, 3 June, 2013 9:00 am to 12:00 noon

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, …, ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

State and prove Gauss's Lemma for the Legendre symbol $\left(\frac{a}{p}\right)$. For which odd primes p is 2 a quadratic residue modulo p? Justify your answer.

2F Topics in Analysis

Show that $\sin(1)$ is irrational. [The angle is measured in radians.]

3G Geometry and Groups

Show that any pair of lines in hyperbolic 3-space that does not have a common endpoint must have a common normal. Is this still true when the pair of lines does have a common endpoint?

4H Coding and Cryptography

A binary Huffman code is used for encoding symbols $1, \ldots, m$ occurring with respective probabilities $p_1 \geqslant \cdots \geqslant p_m > 0$ where $\sum_{1 \leqslant j \leqslant m} p_j = 1$. Let s_1 be the length of a shortest codeword and s_m the length of a longest codeword. Determine the maximal and minimal values of each of s_1 and s_m, and find binary trees for which they are attained.

5J Statistical Modelling

Variables $Y_1, \ldots, Y_n$ are independent, with Y_i having a density $p(y \mid \mu_i)$ governed by an unknown parameter μ_i. Define the *deviance* for a model M that imposes relationships between the (μ_i).

From this point on, suppose $Y_i \sim \text{Poisson}(\mu_i)$. Write down the log-likelihood of data $y_1, \ldots, y_n$ as a function of $\mu_1, \ldots, \mu_n$.

Let $\widehat{\mu}_i$ be the maximum likelihood estimate of μ_i under model M. Show that the deviance for this model is given by

$$2 \sum_{i=1}^{n} \left\{ y_i \log \frac{y_i}{\widehat{\mu}_i} - (y_i - \widehat{\mu}_i) \right\} .$$

Now suppose that, under M, $\log \mu_i = \beta^{\mathrm{T}} x_i$, $i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are known p-dimensional explanatory variables and β is an unknown p-dimensional parameter. Show that $\widehat{\mu} := (\widehat{\mu}_1, \ldots, \widehat{\mu}_n)^{\mathrm{T}}$ satisfies $X^{\mathrm{T}} y = X^{\mathrm{T}} \widehat{\mu}$, where $y = (y_1, \ldots, y_n)^{\mathrm{T}}$ and X is the $(n \times p)$ matrix with rows $x_1^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}$, and express this as an equation for the maximum likelihood estimate $\widehat{\beta}$ of β. [You are not required to solve this equation.]

6A Mathematical Biology

In a discrete-time model, a proportion μ of mature bacteria divides at each time step. When a mature bacterium divides it is destroyed and two new immature bacteria are produced. A proportion λ of the immature bacteria matures at each time step, and a proportion k of mature bacteria dies at each time step. Show that this model may be represented by the equations

$$\begin{aligned} a_{t+1} &= a_t + 2\mu b_t - \lambda a_t \, , \\ b_{t+1} &= b_t - \mu b_t + \lambda a_t - k b_t \, . \end{aligned}$$

Give an expression for the general solution to these equations and show that the population may grow if $\mu > k$.

At time T, the population is treated with an antibiotic that completely stops bacteria from maturing, but otherwise has no direct effects. Explain what will happen to the population of bacteria afterwards, and give expressions for a_t and b_t for $t > T$ in terms of a_T, b_T, μ and k.

7C Dynamical Systems

Consider the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$ which has a hyperbolic fixed point at the origin.

Define the stable and unstable invariant subspaces of the system linearised about the origin. Give a constraint on the dimensions of these two subspaces.

Define the local stable and unstable manifolds of the origin for the system. How are these related to the invariant subspaces of the linearised system?

For the system

$$\begin{aligned} \dot{x} &= -x + x^2 + y^2 , \\ \dot{y} &= y + y^2 - x^2 , \end{aligned}$$

calculate the stable and unstable manifolds of the origin, each correct up to and including cubic order.

8E Further Complex Methods

Prove that there are no second order linear ordinary homogeneous differential equations for which all points in the extended complex plane are analytic.

Find all such equations which have one regular singular point at $z = 0$.

9B Classical Dynamics

Consider an n-dimensional dynamical system with generalized coordinates and momenta (q_i, p_i), $i = 1, 2, ..., n$.

(a) Define the Poisson bracket $\{f, g\}$ of two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$.

(b) Assuming Hamilton's equations of motion, prove that if a function $G(q_i, p_i)$ Poisson commutes with the Hamiltonian, that is $\{G, H\} = 0$, then G is a constant of the motion.

(c) Assume that q_j is an ignorable coordinate, that is the Hamiltonian does not depend on it explicitly. Using the formalism of Poisson brackets prove that the conjugate momentum p_j is conserved.

10D Cosmology

The Friedmann equation and the fluid conservation equation for a closed isotropic and homogeneous cosmology are given by

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{1}{a^2}\,,$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0\,,$$

where the speed of light is set equal to unity, G is the gravitational constant, $a(t)$ is the expansion scale factor, ρ is the fluid mass density and P is the fluid pressure, and overdots denote differentiation with respect to the time coordinate t.

If the universe contains only blackbody radiation and $a = 0$ defines the zero of time t, show that

$$a^2(t) = t(t_* - t)\,,$$

where t_* is a constant. What is the physical significance of the time t_*? What is the value of the ratio $a(t)/t$ at the time when the scale factor is largest? Sketch the curve of $a(t)$ and identify its geometric shape.

Briefly comment on whether this cosmological model is a good description of the observed universe at any time in its history.

SECTION II

11G Geometry and Groups

Define the *modular group* Γ acting on the upper half-plane.

Describe the set S of points z in the upper half-plane that have $\mathrm{Im}(T(z)) \leqslant \mathrm{Im}(z)$ for each $T \in \Gamma$. Hence find a fundamental set for Γ acting on the upper half-plane.

Let A and J be the two Möbius transformations

$$A : z \mapsto z + 1 \qquad \text{and} \qquad J : z \mapsto -1/z \ .$$

When is $\mathrm{Im}(J(z)) > \mathrm{Im}(z)$?

For any point z in the upper half-plane, show that either $z \in S$ or else there is an integer k with

$$\mathrm{Im}(J(A^k(z))) > \mathrm{Im}(z) \ .$$

Deduce that the modular group is generated by A and J.

12H Coding and Cryptography

Define the *bar product* $C_1|C_2$ of binary linear codes C_1 and C_2, where C_2 is a subcode of C_1. Relate the rank and minimum distance of $C_1|C_2$ to those of C_1 and C_2 and justify your answer. Show that if $C^\perp$ denotes the dual code of C, then

$$(C_1|C_2)^\perp = C_2^\perp|C_1^\perp \ .$$

Using the bar product construction, or otherwise, define the Reed–Muller code $\mathrm{RM}(d, r)$ for $0 \leqslant r \leqslant d$. Show that if $0 \leqslant r \leqslant d-1$, then the dual of $\mathrm{RM}(d, r)$ is again a Reed–Muller code.

13J Statistical Modelling

A cricket ball manufacturing company conducts the following experiment. Every day, a bowling machine is set to one of three levels, "Medium", "Fast" or "Spin", and then bowls 100 balls towards the stumps. The number of times the ball hits the stumps and the average wind speed (in kilometres per hour) during the experiment are recorded, yielding the following data (abbreviated):

Day	Wind	Level	Stumps
1	10	Medium	26
2	8	Medium	37
⋮	⋮	⋮	⋮
50	12	Medium	32
51	7	Fast	31
⋮	⋮	⋮	⋮
120	3	Fast	28
121	5	Spin	35
⋮	⋮	⋮	⋮
150	6	Spin	31

Write down a reasonable model for $Y_1, \ldots, Y_{150}$, where Y_i is the number of times the ball hits the stumps on the i^{th} day. Explain briefly why we might want to include interactions between the variables. Write `R` code to fit your model.

The company's statistician fitted her own generalized linear model using `R`, and obtained the following summary (abbreviated):

```
>summary(ball)
Coefficients:
                 Estimate  Std. Error  z value  Pr(>|z|)
 (Intercept)     -0.37258     0.05388   -6.916  4.66e-12  ***
 Wind             0.09055     0.01595    5.676  1.38e-08  ***
 LevelFast       -0.10005     0.08044   -1.244  0.213570
 LevelSpin        0.29881     0.08268    3.614  0.000301  ***
 Wind:LevelFast   0.03666     0.02364    1.551  0.120933
 Wind:LevelSpin  -0.07697     0.02845   -2.705  0.006825  **
```

Why are `LevelMedium` and `Wind:LevelMedium` not listed?

Suppose that, on another day, the bowling machine is set to "Spin", and the wind speed is 5 kilometres per hour. What linear function of the parameters should the statistician use in constructing a predictor of the number of times the ball hits the stumps that day?

Based on the above output, how might you improve the model? How could you fit your new model in `R`?

14E Further Complex Methods

Show that the equation

$$(z-1)w'' - zw' + (4-2z)w = 0$$

has solutions of the form $w(z) = \int_\gamma \exp(zt) f(t) dt$, where

$$f(t) = \frac{\exp(-t)}{(t-a)(t-b)^2}$$

and the contour γ is any closed curve in the complex plane, where a and b are real constants which should be determined.

Use this to find the general solution, evaluating the integrals explicitly.

15D Cosmology

A spherically symmetric star of total mass M_s has pressure $P(r)$ and mass density $\rho(r)$, where r is the radial distance from its centre. These quantities are related by the equations of hydrostatic equilibrium and mass conservation:

$$\begin{aligned} \frac{dP}{dr} &= -\frac{GM(r)\rho}{r^2}\,, \\ \frac{dM}{dr} &= 4\pi\rho r^2\,, \end{aligned}$$

where $M(r)$ is the mass inside radius r.

By integrating from the centre of the star at $r = 0$, where $P = P_c$, to the surface of the star at $r = R_s$, where $P = P_s$, show that

$$4\pi R_s^3 P_s = \Omega + 3\int_0^{M_s} \frac{P}{\rho}\, dM\,,$$

where Ω is the total gravitational potential energy. Show that

$$-\Omega > \frac{GM_s^2}{2R_s}\,.$$

If the surface pressure is negligible and the star is a perfect gas of particles of mass m with number density n and $P = nk_BT$ at temperature T, and radiation pressure can be ignored, then show that

$$3\int_0^{M_s} \frac{P}{\rho}\, dM = \frac{3k_B}{m}\bar{T}\,,$$

where $\bar{T}$ is the mean temperature of the star, which you should define.

Hence, show that the mean temperature of the star satisfies the inequality

$$\bar{T} > \frac{GM_s m}{6k_B R_s}\,.$$

16G Logic and Set Theory

Write down the recursive definitions of ordinal addition, multiplication and exponentiation.

Given that $F\colon \mathbf{On} \to \mathbf{On}$ is a strictly increasing function-class (i.e. $\alpha < \beta$ implies $F(\alpha) < F(\beta)$), show that $\alpha \leqslant F(\alpha)$ for all α.

Show that every ordinal α has a unique representation in the form

$$\alpha = \omega^{\alpha_1}.a_1 + \omega^{\alpha_2}.a_2 + \cdots + \omega^{\alpha_n}.a_n \ ,$$

where $n \in \omega$, $\alpha \geqslant \alpha_1 > \alpha_2 > \cdots > \alpha_n$, and $a_1, a_2, \ldots, a_n \in \omega \setminus \{0\}$.

Under what conditions can an ordinal α be represented in the form

$$\omega^{\beta_1}.b_1 + \omega^{\beta_2}.b_2 + \cdots + \omega^{\beta_m}.b_m \ ,$$

where $\beta_1 < \beta_2 < \cdots < \beta_m$ and $b_1, b_2, \ldots, b_m \in \omega \setminus \{0\}$? Justify your answer.

[The laws of ordinal arithmetic (associative, distributive, etc.) may be assumed without proof.]

17F Graph Theory

State and prove Hall's theorem about matchings in bipartite graphs.

Show that a regular bipartite graph has a matching meeting every vertex.

A graph is *almost r-regular* if each vertex has degree $r-1$ or r. Show that, if $r \geqslant 2$, an almost r-regular graph G must contain an almost $(r-1)$-regular graph H with $V(H) = V(G)$.

[*Hint: First, if possible, remove edges from G whilst keeping it almost r-regular.*]

18I Galois Theory

(i) Give an example of a field F, contained in $\mathbb{C}$, such that X^4+1 is a product of two irreducible quadratic polynomials in $F[X]$. Justify your answer.

(ii) Let F be any extension of degree 3 over $\mathbb{Q}$. Prove that the polynomial X^4+1 is irreducible over F.

(iii) Give an example of a prime number p such that X^4+1 is a product of two irreducible quadratic polynomials in $\mathbb{F}_p[X]$. Justify your answer.

[Standard facts on fields, extensions, and finite fields may be quoted without proof, as long as they are stated clearly.]

19G Representation Theory

State and prove Maschke's Theorem for complex representations of finite groups.

Without using character theory, show that every irreducible complex representation of the dihedral group of order 10, D_{10}, has dimension at most two. List the irreducible complex representations of D_{10} up to isomorphism.

Let V be the set of vertices of a regular pentagon with the usual action of D_{10}. Explicitly decompose the permutation representation $\mathbb{C}V$ into a direct sum of irreducible subrepresentations.

20H Number Fields

Let $f \in \mathbb{Z}[X]$ be a monic irreducible polynomial of degree n. Let $K = \mathbb{Q}(\alpha)$, where α is a root of f.

(i) Show that if $\mathrm{disc}(f)$ is square-free then $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

(ii) In the case $f(X) = X^3 - 3X - 25$ find the minimal polynomial of $\beta = 3/(1-\alpha)$ and hence compute the discriminant of K. What is the index of $\mathbb{Z}[\alpha]$ in $\mathcal{O}_K$?
[Recall that the discriminant of $X^3 + pX + q$ is $-4p^3 - 27q^2$.]

21G Algebraic Topology

(i) Define the notion of the fundamental group $\pi_1(X, x_0)$ of a path-connected space X with base point x_0.

(ii) Prove that if a group G acts freely and properly discontinuously on a simply connected space Z, then $\pi_1(G\backslash Z, x_0)$ is isomorphic to G. [You may assume the homotopy lifting property, provided that you state it clearly.]

(iii) Suppose that p, q are distinct points on the 2-sphere S^2 and that $X = S^2/(p \sim q)$. Exhibit a simply connected space Z with an action of a group G as in (ii) such that $X = G\backslash Z$, and calculate $\pi_1(X, x_0)$.

22F Linear Analysis

State and prove the Closed Graph Theorem. [You may assume any version of the Baire Category Theorem provided it is clearly stated. If you use any other result from the course, then you must prove it.]

Let X be a closed subspace of ℓ_∞ such that X is also a subset of ℓ_1. Show that the left-shift $L\colon X \to \ell_1$, given by $L(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$, is bounded when X is equipped with the sup-norm.

23I Riemann Surfaces

(i) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence r in $(0, \infty)$. Show that there is at least one point a on the circle $C = \{z \in \mathbb{C} : |z| = r\}$ which is a singular point of f, that is, there is no direct analytic continuation of f in any neighbourhood of a.

(ii) Let X and Y be connected Riemann surfaces. Define the space $\mathcal{G}$ of germs of function elements of X into Y. Define the natural topology on $\mathcal{G}$ and the natural map $\pi\colon \mathcal{G} \to X$. [You may assume without proof that the topology on $\mathcal{G}$ is Hausdorff.] Show that π is continuous. Define the natural complex structure on $\mathcal{G}$ which makes it into a Riemann surface. Finally, show that there is a bijection between the connected components of $\mathcal{G}$ and the complete holomorphic functions of X into Y.

24H Algebraic Geometry

Let $V \subset \mathbb{A}^n$ be an affine variety over an algebraically closed field k. What does it mean to say that V is *irreducible*? Show that any non-empty affine variety $V \subset \mathbb{A}^n$ is the union of a finite number of irreducible affine varieties $V_j \subset \mathbb{A}^n$.

Define the *ideal* $I(V)$ of V. Show that $I(V)$ is a prime ideal if and only if V is irreducible.

Assume that the base field k has characteristic zero. Determine the irreducible components of

$$V\big(X_1X_2,\, X_1X_3 + X_2^2 - 1,\, X_1^2(X_1 - X_3)\big) \subset \mathbb{A}^3 .$$

25H Differential Geometry

For $f : X \to Y$ a smooth map of manifolds, define the concepts of *critical point*, *critical value* and *regular value*.

With the obvious identification of $\mathbf{C}$ with $\mathbf{R}^2$, and hence also of $\mathbf{C}^3$ with $\mathbf{R}^6$, show that the complex-valued polynomial $z_1^3 + z_2^2 + z_3^2$ determines a smooth map $f : \mathbf{R}^6 \to \mathbf{R}^2$ whose only critical point is at the origin. Hence deduce that $V := f^{-1}((0,0)) \setminus \{\mathbf{0}\} \subset \mathbf{R}^6$ is a 4-dimensional manifold, and find the equations of its tangent space at any given point $(z_1, z_2, z_3) \in V$.

Now let $S^5 \subset \mathbf{C}^3 = \mathbf{R}^6$ be the unit 5-sphere, defined by $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. Given a point $P = (z_1, z_2, z_3) \in S^5 \cap V$, by considering the vector $(2z_1, 3z_2, 3z_3) \in \mathbf{C}^3 = \mathbf{R}^6$ or otherwise, show that not all tangent vectors to V at P are tangent to S^5. Deduce that $S^5 \cap V \subset \mathbf{R}^6$ is a compact three-dimensional manifold.

[Standard results may be quoted without proof if stated carefully.]

26K Probability and Measure

State Dynkin's π-system/d-system lemma.

Let μ and ν be probability measures on a measurable space $(E, \mathcal{E})$. Let $\mathcal{A}$ be a π-system on E generating $\mathcal{E}$. Suppose that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$. Show that $\mu = \nu$.

What does it mean to say that a sequence of random variables is independent?

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, all uniformly distributed on $[0,1]$. Let Y be another random variable, independent of $(X_n : n \in \mathbb{N})$. Define random variables Z_n in $[0,1]$ by $Z_n = (X_n + Y) \bmod 1$. What is the distribution of Z_1? Justify your answer.

Show that the sequence of random variables $(Z_n : n \in \mathbb{N})$ is independent.

27J Applied Probability

Let $(X_t, t \geqslant 0)$ be a Markov chain on $\{0, 1, \ldots\}$ with Q-matrix given by

$$\begin{aligned} q_{n,n+1} &= \lambda_n \ , \\ q_{n,0} &= \lambda_n \varepsilon_n \quad (n > 0) \ , \\ q_{n,m} &= 0 \quad \text{if } m \notin \{0, n, n+1\} \, , \end{aligned}$$

where $\varepsilon_n, \lambda_n > 0$.

(i) Show that X is transient if and only if $\sum_n \varepsilon_n < \infty$. [You may assume without proof that $x(1-\delta) \leqslant \log(1+x) \leqslant x$ for all $\delta > 0$ and all sufficiently small positive x.]

(ii) Assume that $\sum_n \varepsilon_n < \infty$. Find a necessary and sufficient condition for X to be almost surely explosive. [You may assume without proof standard results about pure birth processes, provided that they are stated clearly.]

(iii) Find a stationary measure for X. For the case $\lambda_n = \lambda$ and $\varepsilon_n = \alpha/(n+1)$ $(\lambda, \alpha > 0)$, show that X is positive recurrent if and only if $\alpha > 1$.

28K Principles of Statistics

When the real parameter Θ takes value θ, variables $X_1, X_2, \ldots$ arise independently from a distribution P_θ having density function $p_\theta(x)$ with respect to an underlying measure μ. Define the *score variable* $U_n(\theta)$ and the *information function* $I_n(\theta)$ for estimation of Θ based on $\boldsymbol{X}^n := (X_1, \ldots, X_n)$, and relate $I_n(\theta)$ to $i(\theta) := I_1(\theta)$.

State and prove the Cramér–Rao inequality for the variance of an unbiased estimator of Θ. Under what conditions does this inequality become an equality? What is the form of the estimator in this case? [You may assume $\mathbb{E}_\theta\{U_n(\theta)\} = 0$, $\text{var}_\theta\{U_n(\theta)\} = I_n(\theta)$, and any further required regularity conditions, without comment.]

Let $\widehat{\Theta}_n$ be the maximum likelihood estimator of Θ based on $\boldsymbol{X}^n$. What is the asymptotic distribution of $n^{\frac{1}{2}}(\widehat{\Theta}_n - \Theta)$ when $\Theta = \theta$?

Suppose that, for each n, $\widehat{\Theta}_n$ is unbiased for Θ, and the variance of $n^{\frac{1}{2}}(\widehat{\Theta}_n - \Theta)$ is exactly equal to its asymptotic variance. By considering the estimator $\alpha\widehat{\Theta}_k + (1-\alpha)\widehat{\Theta}_n$, or otherwise, show that, for $k < n$, $\text{cov}_\theta(\widehat{\Theta}_k, \widehat{\Theta}_n) = \text{var}_\theta(\widehat{\Theta}_n)$.

29J Stochastic Financial Models

(i) Suppose that the price S_t of an asset at time t is given by

$$S_t = S_0 \exp\{\, \sigma B_t + (r - \tfrac{1}{2}\sigma^2)t \,\},$$

where B is a Brownian motion, S_0 and σ are positive constants, and r is the riskless rate of interest, assumed constant. In this model, explain briefly why the time-0 price of a derivative which delivers a bounded random variable Y at time T should be given by $E(e^{-rT}Y)$. What feature of this model ensures that the price is unique?

Derive an expression $C(S_0, K, T, r, \sigma)$ for the time-0 price of a *European call option* with *strike* K and *expiry* T. Explain the italicized terms.

(ii) Suppose now that the price X_t of an asset at time t is given by

$$X_t = \sum_{j=1}^{n} w_j \, \exp\{\, \sigma_j B_t + (r - \tfrac{1}{2}\sigma_j^2)t \,\},$$

where the w_j and σ_j are positive constants, and the other notation is as in part (i) above. Show that the time-0 price of a European call option with strike K and expiry T written on this asset can be expressed as

$$\sum_{j=1}^{n} C(w_j, k_j, T, r, \sigma_j),$$

where the k_j are constants. Explain how the k_j are characterized.

30C Partial Differential Equations

(i) Discuss briefly the concept of *well-posedness* of a Cauchy problem for a partial differential equation.

Solve the Cauchy problem

$$\partial_2 u + x_1 \partial_1 u = a u^2 \,, \qquad u(x_1, 0) = \phi(x_1) \,,$$

where $a \in \mathbb{R}$, $\phi \in C^1(\mathbb{R})$ and ∂_i denotes the partial derivative with respect to x_i for $i = 1, 2$.

For the case $a = 0$ show that the solution satisfies $\max_{x_1 \in \mathbb{R}} |u(x_1, x_2)| = \|\phi\|_{C^0}$, where the C^r norm on functions $\phi = \phi(x_1)$ of one variable is defined by

$$\|\phi\|_{C^r} = \sum_{i=0}^{r} \max_{x \in \mathbb{R}} |\partial_1^i \phi(x_1)|.$$

Deduce that the Cauchy problem is then well-posed in the uniform metric (i.e. the metric determined by the C^0 norm).

(ii) State the Cauchy–Kovalevskaya theorem and deduce that the following Cauchy problem for the Laplace equation,

$$\partial_1^2 u + \partial_2^2 u = 0 \,, \qquad u(x_1, 0) = 0 \,,\ \partial_2 u(x_1, 0) = \phi(x_1) \,, \qquad (*)$$

has a unique analytic solution in some neighbourhood of $x_2 = 0$ for any analytic function $\phi = \phi(x_1)$. Write down the solution for the case $\phi(x_1) = \sin(n x_1)$, and hence give a sequence of initial data $\{\phi_n(x_1)\}_{n=1}^{\infty}$ with the property that

$$\|\phi_n\|_{C^r} \;\to\; 0 \,, \quad \text{as } n \to \infty, \text{ for each } \; r \in \mathbb{N} \,,$$

whereas u_n, the corresponding solution of $(*)$, satisfies

$$\max_{x_1 \in \mathbb{R}} |u_n(x_1, x_2)| \;\to\; +\infty \,, \quad \text{as } n \to \infty,$$

for any $x_2 \neq 0$.

31B Asymptotic Methods

Suppose $\alpha > 0$. Define what it means to say that

$$F(x) \sim \frac{1}{\alpha x} \sum_{n=0}^{\infty} n! \left(\frac{-1}{\alpha x}\right)^n$$

is an asymptotic expansion of $F(x)$ as $x \to \infty$. Show that $F(x)$ has no other asymptotic expansion in inverse powers of x as $x \to \infty$.

To estimate the value of $F(x)$ for large x, one may use an *optimal truncation* of the asymptotic expansion. Explain what is meant by this, and show that the error is an exponentially small quantity in x.

Derive an integral respresentation for a function $F(x)$ with the above asymptotic expansion.

32C Integrable Systems

Quoting carefully all necessary results, use the theory of inverse scattering to derive the 1-soliton solution of the KdV equation

$$u_t = 6uu_x - u_{xxx}\,.$$

33E Principles of Quantum Mechanics

Consider a composite system of several identical particles. Describe how the multi-particle state is constructed from single-particle states. For the case of two identical particles, describe how considering the interchange symmetry leads to the definition of bosons and fermions.

Consider two non-interacting, identical particles, each with spin 1. The single-particle, spin-independent Hamiltonian $H(\hat{\mathbf{x}}_i, \hat{\mathbf{p}}_i)$ has non-degenerate eigenvalues E_n and wavefunctions $\psi_n(\mathbf{x}_i)$ where $i = 1, 2$ labels the particle and $n = 0, 1, 2, 3, \ldots$. In terms of these single-particle wavefunctions and single-particle spin states $|1\rangle$, $|0\rangle$ and $|-1\rangle$, write down all of the two-particle states and energies for:

(i) the ground state;

(ii) the first excited state.

Assume now that E_n is a linear function of n. Find the degeneracy of the N^{th} energy level of the two-particle system for:

(iii) N even;

(iv) N odd.

34D Applications of Quantum Mechanics

Consider a quantum system with Hamiltonian $\widehat{H}$ and energy levels

$$E_0 < E_1 < E_2 < \dots \,.$$

For any state $|\psi\rangle$ define the *Rayleigh–Ritz quotient* $R[\psi]$ and show the following:

(i) the ground state energy E_0 is the minimum value of $R[\psi]$;

(ii) all energy eigenstates are stationary points of $R[\psi]$ with respect to variations of $|\psi\rangle$.

Under what conditions can the value of $R[\psi_\alpha]$ for a trial wavefunction ψ_α (depending on some parameter α) be used as an estimate of the energy E_1 of the first excited state? Explain your answer.

For a suitably chosen trial wavefunction which is the product of a polynomial and a Gaussian, use the Rayleigh–Ritz quotient to estimate E_1 for a particle of mass m moving in a potential $V(x) = g|x|$, where g is a constant.

[You may use the integral formulae,

$$\int_0^\infty x^{2n} \exp\left(-px^2\right) dx = \frac{(2n-1)!!}{2(2p)^n} \sqrt{\frac{\pi}{p}}$$

$$\int_0^\infty x^{2n+1} \exp\left(-px^2\right) dx = \frac{n!}{2p^{n+1}}$$

where n is a non-negative integer and p is a constant.]

35A Statistical Physics

(i) What is the occupation number of a state i with energy E_i according to the Fermi–Dirac statistics for a given chemical potential μ?

(ii) Assuming that the energy E is spin independent, what is the number g_s of electrons which can occupy an energy level?

(iii) Consider a semi-infinite metal slab occupying $z \leqslant 0$ (and idealized to have infinite extent in the xy plane) and a vacuum environment at $z > 0$. An electron with momentum (p_x, p_y, p_z) inside the slab will escape the metal in the $+z$ direction if it has a sufficiently large momentum p_z to overcome a potential barrier V_0 relative to the Fermi energy ϵ_{F}, i.e. if

$$\frac{p_z^2}{2m} \geqslant \epsilon_{\mathrm{F}} + V_0 \,,$$

where m is the electron mass.

At fixed temperature T, some fraction of electrons will satisfy this condition, which results in a current density j_z in the $+z$ direction (an electron having escaped the metal once is considered lost, never to return). Each electron escaping provides a contribution $\delta j_z = -ev_z$ to this current density, where v_z is the velocity and e the elementary charge.

(a) Briefly describe the Fermi–Dirac distribution as a function of energy in the limit $k_{\mathrm{B}}T \ll \epsilon_{\mathrm{F}}$, where k_{B} is the Boltzmann constant. What is the chemical potential μ in this limit?

(b) Assume that the electrons behave like an ideal, non-relativistic Fermi gas and that $k_{\mathrm{B}}T \ll V_0$ and $k_{\mathrm{B}}T \ll \epsilon_{\mathrm{F}}$. Calculate the current density j_z associated with the electrons escaping the metal in the $+z$ direction. How could we easily increase the strength of the current?

36B Electrodynamics

(i) Starting from

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0 \end{pmatrix}$$

and performing a Lorentz transformation with $\gamma = 1/\sqrt{1-u^2/c^2}$, using

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma u/c & 0 & 0 \\ -\gamma u/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

show how $\mathbf{E}$ and $\mathbf{B}$ transform under a Lorentz transformation.

(ii) By taking the limit $c \to \infty$, obtain the behaviour of $\mathbf{E}$ and $\mathbf{B}$ under a Galilei transfomation and verify the invariance under Galilei transformations of the non-relativistic equation

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,.$$

(iii) Show that Maxwell's equations admit solutions of the form

$$\mathbf{E} = \mathbf{E}_0\, f(t - \mathbf{n}\cdot\mathbf{x}/c)\,, \qquad \mathbf{B} = \mathbf{B}_0\, f(t - \mathbf{n}\cdot\mathbf{x}/c)\,, \tag{$\star$}$$

where f is an arbitrary function, $\mathbf{n}$ is a unit vector, and the constant vectors $\mathbf{E}_0$ and $\mathbf{B}_0$ are subject to restrictions which should be stated.

(iv) Perform a Galilei transformation of a solution ($\star$), with $\mathbf{n} = (1, 0, 0)$. Show that, by a particular choice of u, the solution may brought to the form

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 g(\tilde{x})\,, \qquad \tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0 g(\tilde{x})\,, \tag{$\dagger$}$$

where g is an arbitrary function and $\tilde{x}$ is a spatial coordinate in the rest frame. By showing that ($\dagger$) is not a solution of Maxwell's equations in the boosted frame, conclude that Maxwell's equations are not invariant under Galilei transformations.

37D General Relativity

The curve γ, $x^a = x^a(\lambda)$, is a geodesic with affine parameter λ. Write down the geodesic equation satisfied by $x^a(\lambda)$.

Suppose the parameter is changed to $\mu(\lambda)$, where $d\mu/d\lambda > 0$. Obtain the corresponding equation and find the condition for μ to be affine. Deduce that, whatever parametrization ν is used along the curve γ, the tangent vector K^a to γ satisfies

$$(\nabla_\nu K)^{[a} K^{b]} = 0\,.$$

Now consider a spacetime with metric g_{ab}, and conformal transformation

$$\tilde{g}_{ab} = \Omega^2(x^c) g_{ab}\,.$$

The curve γ is a geodesic of the metric connection of g_{ab}. What further restriction has to be placed on γ so that it is also a geodesic of the metric connection of $\tilde{g}_{ab}$? Justify your answer.

38A Fluid Dynamics II

The velocity field $\mathbf{u}$ and stress tensor σ satisfy the Stokes equations in a volume V bounded by a surface S. Let $\hat{\mathbf{u}}$ be another solenoidal velocity field. Show that

$$\int_S \sigma_{ij} n_j \hat{u}_i \, dS = \int_V 2\mu e_{ij} \hat{e}_{ij} \, dV\,,$$

where $\mathbf{e}$ and $\hat{\mathbf{e}}$ are the strain-rates corresponding to the velocity fields $\mathbf{u}$ and $\hat{\mathbf{u}}$ respectively, and $\mathbf{n}$ is the unit normal vector out of V. Hence, or otherwise, prove the minimum dissipation theorem for Stokes flow.

A particle moves at velocity $\mathbf{U}$ through a highly viscous fluid of viscosity μ contained in a stationary vessel. As the particle moves, the fluid exerts a drag force $\mathbf{F}$ on it. Show that

$$-\mathbf{F}\cdot\mathbf{U} = \int_V 2\mu e_{ij} e_{ij} \, dV\,.$$

Consider now the case when the particle is a small cube, with sides of length ℓ, moving in a very large vessel. You may assume that

$$\mathbf{F} = -k\mu\ell\mathbf{U}\,,$$

for some constant k. Use the minimum dissipation theorem, being careful to declare the domain(s) involved, to show that

$$3\pi \leqslant k \leqslant 3\sqrt{3}\pi.$$

[You may assume Stokes' result for the drag on a sphere of radius a, $\mathbf{F} = -6\pi\mu a\mathbf{U}$.]

39C Waves

Starting from the equations for the one-dimensional unsteady flow of a perfect gas of uniform entropy, show that the Riemann invariants

$$R_{\pm} = u \pm \frac{2}{\gamma - 1}(c - c_0)$$

are constant on characteristics $C_{\pm}$ given by $dx/dt = u \pm c$, where $u(x,t)$ is the velocity of the gas, $c(x,t)$ is the local speed of sound, c_0 is a constant and γ is the ratio of specific heats.

Such a gas initially occupies the region $x > 0$ to the right of a piston in an infinitely long tube. The gas and the piston are initially at rest with $c = c_0$. At time $t = 0$ the piston starts moving to the left at a constant velocity V. Find $u(x,t)$ and $c(x,t)$ in the three regions

$$\begin{array}{ll} \text{(i)} & c_0 t \leqslant x\,, \\ \text{(ii)} & at \leqslant x \leqslant c_0 t\,, \\ \text{(iii)} & -Vt \leqslant x \leqslant at\,, \end{array}$$

where $a = c_0 - \frac{1}{2}(\gamma+1)V$. What is the largest value of V for which c is positive throughout region (iii)? What happens if V exceeds this value?

40C Numerical Analysis

Let

$$A(\alpha) = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}\,.$$

(i) For which values of α is $A(\alpha)$ positive definite?

(ii) Formulate the Gauss–Seidel method for the solution $\mathbf{x} \in \mathbb{R}^3$ of a system

$$A(\alpha)\mathbf{x} = \mathbf{b}\,,$$

with $A(\alpha)$ as defined above and $\mathbf{b} \in \mathbb{R}^3$. Prove that the Gauss–Seidel method converges to the solution of the above system whenever A is positive definite. [You may state and use the Householder–John theorem without proof.]

(iii) For which values of α does the Jacobi iteration applied to the solution of the above system converge?

END OF PAPER