UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 7 June, 2013 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

*Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles labelled **A, B, ..., H** according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

What is a quadratic form on a finite dimensional real vector space V? What does it mean for two quadratic forms to be isomorphic (*i.e.* congruent)? State Sylvester's law of inertia and explain the definition of the quantities which appear in it. Find the signature of the quadratic form on $\mathbb{R}^3$ given by $q(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$, where

$$A = \begin{pmatrix} -2 & 1 & 6 \\ 1 & -1 & -3 \\ 6 & -3 & 1 \end{pmatrix}.$$

2G Groups, Rings and Modules

Let p be a prime number, and G be a non-trivial finite group whose order is a power of p. Show that the size of every conjugacy class in G is a power of p. Deduce that the centre Z of G has order at least p.

3F Analysis II

State and prove the chain rule for differentiable mappings $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^k$.

Suppose now $F : \mathbb{R}^2 \to \mathbb{R}^2$ has image lying on the unit circle in $\mathbb{R}^2$. Prove that the determinant $\det(DF|_x)$ vanishes for every $x \in \mathbb{R}^2$.

4E Complex Analysis

State Rouché's theorem. How many roots of the polynomial $z^8 + 3z^7 + 6z^2 + 1$ are contained in the annulus $1 < |z| < 2$?

5C Methods

Show that the general solution of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$$

can be written in the form

$$y(x,t) = f(ct-x) + g(ct+x)\,.$$

For the boundary conditions

$$y(0,t) = y(L,t) = 0, \qquad t>0\,,$$

find the relation between f and g and show that they are $2L$-periodic. Hence show that

$$E(t) = \frac{1}{2}\int_0^L \left(\frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2\right) dx$$

is independent of t.

6B Quantum Mechanics

The components of the three-dimensional angular momentum operator $\hat{\mathbf{L}}$ are defined as follows:

$$\hat{L}_x = -i\hbar\Big(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\Big) \quad \hat{L}_y = -i\hbar\Big(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\Big) \quad \hat{L}_z = -i\hbar\Big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\Big)\,.$$

Given that the wavefunction

$$\psi = (f(x) + iy)z$$

is an eigenfunction of $\hat{L}_z$, find all possible values of $f(x)$ and the corresponding eigenvalues of ψ. Letting $f(x) = x$, show that ψ is an eigenfunction of $\hat{\mathbf{L}}^2$ and calculate the corresponding eigenvalue.

7D Electromagnetism

The infinite plane $z = 0$ is earthed and the infinite plane $z = d$ carries a charge of σ per unit area. Find the electrostatic potential between the planes.

Show that the electrostatic energy per unit area (of the planes $z =$ constant) between the planes can be written as either $\frac{1}{2}\sigma^2 d/\epsilon_0$ or $\frac{1}{2}\epsilon_0 V^2/d$, where V is the potential at $z = d$.

The distance between the planes is now increased by αd, where α is small. Show that the change in the energy per unit area is $\frac{1}{2}\sigma V\alpha$ if the upper plane ($z = d$) is electrically isolated, and is approximately $-\frac{1}{2}\sigma V\alpha$ if instead the potential on the upper plane is maintained at V. Explain briefly how this difference can be accounted for.

8C Numerical Analysis

For a continuous function f, and $k + 1$ distinct points $\{x_0, x_1, \ldots, x_k\}$, define the divided difference $f[x_0, \ldots, x_k]$ of order k.

Given $n + 1$ points $\{x_0, x_1, \ldots, x_n\}$, let $p_n \in \mathbb{P}_n$ be the polynomial of degree n that interpolates f at these points. Prove that p_n can be written in the Newton form

$$p_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i) .$$

9H Markov Chains

Suppose P is the transition matrix of an irreducible recurrent Markov chain with state space I. Show that if x is an invariant measure and $x_k > 0$ for some $k \in I$, then $x_j > 0$ for all $j \in I$.

Let

$$\gamma_j^k = p_{kj} + \sum_{t=1}^{\infty} \sum_{i_1 \neq k, \ldots, i_t \neq k} p_{ki_t} p_{i_t i_{t-1}} \cdots p_{i_1 j} .$$

Give a meaning to γ_j^k and explain why $\gamma_k^k = 1$.

Suppose x is an invariant measure with $x_k = 1$. Prove that $x_j \geqslant \gamma_j^k$ for all j.

SECTION II

10E Linear Algebra

What does it mean for an $n \times n$ matrix to be in Jordan form? Show that if $A \in M_{n\times n}(\mathbb{C})$ is in Jordan form, there is a sequence (A_m) of diagonalizable $n \times n$ matrices which converges to A, in the sense that the (ij)th component of A_m converges to the (ij)th component of A for all i and j. [*Hint: A matrix with distinct eigenvalues is diagonalizable.*] Deduce that the same statement holds for all $A \in M_{n\times n}(\mathbb{C})$.

Let $V = M_{2\times 2}(\mathbb{C})$. Given $A \in V$, define a linear map $T_A : V \to V$ by $T_A(B) = AB + BA$. Express the characteristic polynomial of T_A in terms of the trace and determinant of A. [*Hint: First consider the case where A is diagonalizable.*]

11G Groups, Rings and Modules

Let R be an integral domain, and M be a finitely generated R-module.

(i) Let S be a finite subset of M which generates M as an R-module. Let T be a maximal linearly independent subset of S, and let N be the R-submodule of M generated by T. Show that there exists a non-zero $r \in R$ such that $rx \in N$ for every $x \in M$.

(ii) Now assume M is *torsion-free*, i.e. $rx = 0$ for $r \in R$ and $x \in M$ implies $r = 0$ or $x = 0$. By considering the map $M \to N$ mapping x to rx for r as in (i), show that every torsion-free finitely generated R-module is isomorphic to an R-submodule of a finitely generated free R-module.

12F Analysis II

State the contraction mapping theorem.

A metric space (X, d) is bounded if $\{d(x, y) \,|\, x, y \in X\}$ is a bounded subset of $\mathbb{R}$. Suppose (X, d) is complete and bounded. Let $\text{Maps}(X, X)$ denote the set of continuous maps from X to itself. For $f, g \in \text{Maps}(X, X)$, let

$$\delta(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

Prove that $(\text{Maps}(X, X), \delta)$ is a complete metric space. Is the subspace $\mathcal{C} \subset \text{Maps}(X, X)$ of contraction mappings a complete subspace?

Let $\tau : \mathcal{C} \to X$ be the map which associates to any contraction its fixed point. Prove that τ is continuous.

13G Metric and Topological Spaces

Let X be a topological space. A *connected component* of X means an equivalence class with respect to the equivalence relation on X defined as:

$$x \sim y \iff x, y \text{ belong to some connected subspace of } X.$$

(i) Show that every connected component is a connected and closed subset of X.

(ii) If X, Y are topological spaces and $X \times Y$ is the product space, show that every connected component of $X \times Y$ is a direct product of connected components of X and Y.

14D Complex Methods

Let C_1 and C_2 be the circles $x^2 + y^2 = 1$ and $5x^2 - 4x + 5y^2 = 0$, respectively, and let D be the (finite) region between the circles. Use the conformal mapping

$$w = \frac{z-2}{2z-1}$$

to solve the following problem:

$$\nabla^2 \phi = 0 \text{ in } D \text{ with } \phi = 1 \text{ on } C_1 \text{ and } \phi = 2 \text{ on } C_2.$$

15F Geometry

Let η be a smooth curve in the xz-plane $\eta(s) = (f(s), 0, g(s))$, with $f(s) > 0$ for every $s \in \mathbb{R}$ and $f'(s)^2 + g'(s)^2 = 1$. Let S be the surface obtained by rotating η around the z-axis. Find the first fundamental form of S.

State the equations for a curve $\gamma : (a, b) \to S$ parametrised by arc-length to be a geodesic.

A parallel on S is the closed circle swept out by rotating a single point of η. Prove that for every $n \in \mathbb{Z}_{>0}$ there is some η for which exactly n parallels are geodesics. Sketch possible such surfaces S in the cases $n = 1$ and $n = 2$.

If *every* parallel is a geodesic, what can you deduce about S ? Briefly justify your answer.

16A Variational Principles

Derive the Euler–Lagrange equation for the integral

$$\int_a^b f(x, y, y', y'')\, dx$$

where prime denotes differentiation with respect to x, and both y and y' are specified at $x = a, b$.

Find $y(x)$ that extremises the integral

$$\int_0^\pi \left(y + \frac{1}{2}y^2 - \frac{1}{2}y''^2 \right) dx$$

subject to $y(0) = -1$, $y'(0) = 0$, $y(\pi) = \cosh \pi$ and $y'(\pi) = \sinh \pi$.

Show that your solution is a global maximum. You may use the result that

$$\int_0^\pi \phi^2(x) dx \leqslant \int_0^\pi \phi'^2(x) dx$$

for any (suitably differentiable) function ϕ which satisfies $\phi(0) = 0$ and $\phi(\pi) = 0$.

17C Methods

Find the inverse Fourier transform $G(x)$ of the function

$$g(k) = e^{-a|k|}, \qquad a > 0, \qquad -\infty < k < \infty\,.$$

Assuming that appropriate Fourier transforms exist, determine the solution $\psi(x, y)$ of

$$\nabla^2 \psi = 0, \qquad -\infty < x < \infty, \qquad 0 < y < 1,$$

with the following boundary conditions

$$\psi(x, 0) = \delta(x), \qquad \psi(x, 1) = \frac{1}{\pi} \frac{1}{x^2 + 1}\,.$$

Here $\delta(x)$ is the Dirac delta-function.

18A Fluid Dynamics

The axisymmetric, irrotational flow generated by a solid sphere of radius a translating at velocity U in an inviscid, incompressible fluid is represented by a velocity potential $\phi(r,\theta)$. Assume the fluid is at rest far away from the sphere. Explain briefly why $\nabla^2\phi = 0$.

By trying a solution of the form $\phi(r,\theta) = f(r)\,g(\theta)$, show that

$$\phi = -\frac{Ua^3\cos\theta}{2r^2}$$

and write down the fluid velocity.

Show that the total kinetic energy of the fluid is $kMU^2/4$ where M is the mass of the sphere and k is the ratio of the density of the fluid to the density of the sphere.

A heavy sphere (i.e. $k < 1$) is released from rest in an inviscid fluid. Determine its speed after it has fallen a distance h in terms of M, k, g and h.

Note, in spherical polars:

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right).$$

19H Statistics

Explain the notion of a sufficient statistic.

Suppose X is a random variable with distribution F taking values in $\{1,\ldots,6\}$, with $P(X=i) = p_i$. Let $x_1,\ldots,x_n$ be a sample from F. Suppose n_i is the number of these x_j that are equal to i. Use a factorization criterion to explain why $(n_1,\ldots,n_6)$ is sufficient for $\theta = (p_1,\ldots,p_6)$.

Let H_0 be the hypothesis that $p_i = 1/6$ for all i. Derive the statistic of the generalized likelihood ratio test of H_0 against the alternative that this is not a good fit.

Assuming that $n_i \approx n/6$ when H_0 is true and n is large, show that this test can be approximated by a chi-squared test using a test statistic

$$T = -n + \frac{6}{n}\sum_{i=1}^{6} n_i^2.$$

Suppose $n = 100$ and $T = 8.12$. Would you reject H_0? Explain your answer.

20H Optimization

Given real numbers a and b, consider the problem P of minimizing

$$f(x) = ax_{11} + 2x_{12} + 3x_{13} + bx_{21} + 4x_{22} + x_{23}$$

subject to $x_{ij} \geqslant 0$ and

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= 5 \\ x_{21} + x_{22} + x_{23} &= 5 \\ x_{11} + x_{21} &= 3 \\ x_{12} + x_{22} &= 3 \\ x_{13} + x_{23} &= 4. \end{aligned}$$

List all the basic feasible solutions, writing them as 2×3 matrices (x_{ij}).

Let $f(x) = \sum_{ij} c_{ij} x_{ij}$. Suppose there exist λ_i, μ_j such that

$$\lambda_i + \mu_j \leqslant c_{ij} \text{ for all } i \in \{1,2\},\ j \in \{1,2,3\}\,.$$

Prove that if x and x' are both feasible for P and $\lambda_i + \mu_j = c_{ij}$ whenever $x_{ij} > 0$, then $f(x) \leqslant f(x')$.

Let x^* be the initial feasible solution that is obtained by formulating P as a transportation problem and using a greedy method that starts in the upper left of the matrix (x_{ij}). Show that if $a + 2 \leqslant b$ then x^* minimizes f.

For what values of a and b is one step of the transportation algorithm sufficient to pivot from x^* to a solution that *maximizes* f?

END OF PAPER