UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 6 June, 2013 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Groups, Rings and Modules

Define the notion of a free module over a ring. When R is a PID, show that every ideal of R is free as an R-module.

2F Analysis II

For each of the following sequences of functions on $[0,1]$, indexed by $n = 1, 2, \ldots$, determine whether or not the sequence has a pointwise limit, and if so, determine whether or not the convergence to the pointwise limit is uniform.

1. $f_n(x) = 1/(1 + n^2x^2)$
2. $g_n(x) = nx(1 - x)^n$
3. $h_n(x) = \sqrt{n}x(1 - x)^n$

3G Metric and Topological Spaces

Let X be a metric space with the metric $d : X \times X \to \mathbb{R}$.

(i) Show that if X is compact as a topological space, then X is complete.

(ii) Show that the completeness of X is not a topological property, i.e. give an example of two metrics d, d' on a set X, such that the associated topologies are the same, but (X, d) is complete and (X, d') is not.

4D Complex Methods

Let $y(t) = 0$ for $t < 0$, and let $\lim_{t \to 0^+} y(t) = y_0$.

(i) Find the Laplace transforms of $H(t)$ and $tH(t)$, where $H(t)$ is the Heaviside step function.

(ii) Given that the Laplace transform of $y(t)$ is $\widehat{y}(s)$, find expressions for the Laplace transforms of $\dot{y}(t)$ and $y(t - 1)$.

(iii) Use Laplace transforms to solve the equation

$$\dot{y}(t) - y(t - 1) = H(t) - (t - 1)H(t - 1)$$

in the case $y_0 = 0$.

5F Geometry

Let S be a surface with Riemannian metric having first fundamental form $du^2 + G(u,v)dv^2$. State a formula for the Gauss curvature K of S.

Suppose that S is flat, so K vanishes identically, and that $u = 0$ is a geodesic on S when parametrised by arc-length. Using the geodesic equations, or otherwise, prove that $G(u,v) \equiv 1$, i.e. S is locally isometric to a plane.

6A Variational Principles

A cylindrical drinking cup has thin curved sides with density ρ per unit area, and a disk-shaped base with density $k\rho$ per unit area. The cup has capacity to hold a fixed volume V of liquid. Use the method of Lagrange multipliers to find the minimum mass of the cup.

7C Methods

The solution to the Dirichlet problem on the half-space $D = \{\mathbf{x} = (x,y,z) : z > 0\}$:

$$\nabla^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in D\,, \qquad u(\mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \qquad u(x,y,0) = h(x,y),$$

is given by the formula

$$u(\mathbf{x}_0) = u(x_0,y_0,z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\frac{\partial}{\partial n}G(\mathbf{x},\mathbf{x}_0)\,dx\,dy\,,$$

where n is the outward normal to ∂D.

State the boundary conditions on G and explain how G is related to G_3, where

$$G_3(\mathbf{x},\mathbf{x}_0) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}_0|}$$

is the fundamental solution to the Laplace equation in three dimensions.

Using the method of images find an explicit expression for the function $\frac{\partial}{\partial n}G(\mathbf{x},\mathbf{x}_0)$ in the formula.

8B Quantum Mechanics

If α, β and γ are linear operators, establish the identity

$$[\alpha\beta, \gamma] = \alpha[\beta, \gamma] + [\alpha, \gamma]\beta.$$

In what follows, the operators x and p are Hermitian and represent position and momentum of a quantum mechanical particle in one-dimension. Show that

$$[x^n, p] = i\hbar n x^{n-1}$$

and

$$[x, p^m] = i\hbar m p^{m-1}$$

where $m, n \in \mathbb{Z}^+$. Assuming $[x^n, p^m] \neq 0$, show that the operators x^n and p^m are Hermitian but their product is not. Determine whether $x^n p^m + p^m x^n$ is Hermitian.

9H Markov Chains

Prove that if a distribution π is in detailed balance with a transition matrix P then it is an invariant distribution for P.

Consider the following model with 2 urns. At each time, $t = 0, 1, \ldots$ one of the following happens:

- with probability β a ball is chosen at random and moved to the other urn (but nothing happens if both urns are empty);
- with probability γ a ball is chosen at random and removed (but nothing happens if both urns are empty);
- with probability α a new ball is added to a randomly chosen urn,

where $\alpha + \beta + \gamma = 1$ and $\alpha < \gamma$. State (i, j) denotes that urns 1, 2 contain i and j balls respectively. Prove that there is an invariant measure

$$\lambda_{i,j} = \frac{(i+j)!}{i!j!}(\alpha/2\gamma)^{i+j}.$$

Find the proportion of time for which there are n balls in the system.

SECTION II

10E Linear Algebra

Let V and W be finite dimensional real vector spaces and let $T : V \to W$ be a linear map. Define the dual space V^* and the dual map T^*. Show that there is an isomorphism $\iota : V \to (V^*)^*$ which is canonical, in the sense that $\iota \circ S = (S^*)^* \circ \iota$ for any automorphism S of V.

Now let W be an inner product space. Use the inner product to show that there is an injective map from $\operatorname{im} T$ to $\operatorname{im} T^*$. Deduce that the row rank of a matrix is equal to its column rank.

11G Groups, Rings and Modules

Let $R = \mathbb{C}[X, Y]$ be the polynomial ring in two variables over the complex numbers, and consider the principal ideal $I = (X^3 - Y^2)$ of R.

(i) Using the fact that R is a UFD, show that I is a prime ideal of R. [*Hint: Elements in* $\mathbb{C}[X, Y]$ *are polynomials in* Y *with coefficients in* $\mathbb{C}[X]$.]

(ii) Show that I is not a maximal ideal of R, and that it is contained in infinitely many distinct proper ideals in R.

12F Analysis II

For each of the following statements, provide a proof or justify a counterexample.

1. The norms $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ and $\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|$ on $\mathbb{R}^n$ are Lipschitz equivalent.
2. The norms $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ and $\|x\|_\infty = \max_i |x_i|$ on the vector space of sequences $(x_i)_{i \geqslant 1}$ with $\sum |x_i| < \infty$ are Lipschitz equivalent.
3. Given a linear function $\phi : V \to W$ between normed real vector spaces, there is some N for which $\|\phi(x)\| \leqslant N$ for every $x \in V$ with $\|x\| \leqslant 1$.
4. Given a linear function $\phi : V \to W$ between normed real vector spaces for which there is some N for which $\|\phi(x)\| \leqslant N$ for every $x \in V$ with $\|x\| \leqslant 1$, then ϕ is continuous.
5. The uniform norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ is complete on the vector space of continuous real-valued functions f on $\mathbb{R}$ for which $f(x) = 0$ for $|x|$ sufficiently large.
6. The uniform norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ is complete on the vector space of continuous real-valued functions f on $\mathbb{R}$ which are bounded.

13E Complex Analysis

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk, and let C be its boundary (the unit circle), with the anticlockwise orientation. Suppose $\phi : C \to \mathbb{C}$ is continuous. Stating clearly any theorems you use, show that

$$g_\phi(w) = \frac{1}{2\pi i} \int_C \frac{\phi(z)}{z - w} dz$$

is an analytic function of w for $w \in D$.

Now suppose ϕ is the restriction of a holomorphic function F defined on some annulus $1 - \epsilon < |z| < 1 + \epsilon$. Show that $g_\phi(w)$ is the restriction of a holomorphic function defined on the open disc $|w| < 1 + \epsilon$.

Let $f_\phi : [0, 2\pi] \to \mathbb{C}$ be defined by $f_\phi(\theta) = \phi(e^{i\theta})$. Express the coefficients in the power series expansion of g_ϕ centered at 0 in terms of f_ϕ.

Let $n \in \mathbb{Z}$. What is g_ϕ in the following cases?

1. $\phi(z) = z^n$.
2. $\phi(z) = \overline{z}^n$.
3. $\phi(z) = (\mathrm{Re}\ z)^2$.

14F Geometry

Show that the set of all straight lines in $\mathbb{R}^2$ admits the structure of an abstract smooth surface S. Show that S is an open Möbius band (i.e. the Möbius band without its boundary circle), and deduce that S admits a Riemannian metric with vanishing Gauss curvature.

Show that there is no metric $d : S \times S \to \mathbb{R}_{\geqslant 0}$, in the sense of metric spaces, which

1. induces the locally Euclidean topology on S constructed above;
2. is invariant under the natural action on S of the group of translations of $\mathbb{R}^2$.

Show that the set of great circles on the two-dimensional sphere admits the structure of a smooth surface S'. Is S' homeomorphic to S ? Does S' admit a Riemannian metric with vanishing Gauss curvature? Briefly justify your answers.

15C Methods

The Laplace equation in plane polar coordinates has the form

$$\nabla^2\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0\,.$$

Using separation of variables, derive the general solution to the equation that is single-valued in the domain $1 < r < 2$.

For

$$f(\theta) = \sum_{n=1}^{\infty} A_n \sin n\theta\,,$$

solve the Laplace equation in the annulus with the boundary conditions:

$$\nabla^2\phi = 0, \quad 1 < r < 2, \qquad \phi(r,\theta) = \begin{cases} f(\theta), & r = 1 \\ f(\theta) + 1, & r = 2. \end{cases}$$

16B Quantum Mechanics

Obtain, with the aid of the time-dependent Schrödinger equation, the conservation equation

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t) + \nabla\cdot\mathbf{j}(\mathbf{x},t) = 0$$

where $\rho(\mathbf{x},t)$ is the probability density and $\mathbf{j}(\mathbf{x},t)$ is the probability current. What have you assumed about the potential energy of the system?

Show that if the potential $U(\mathbf{x},t)$ is complex the conservation equation becomes

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t) + \nabla\cdot\mathbf{j}(\mathbf{x},t) = \frac{2}{\hbar}\rho(\mathbf{x},t)\,\mathrm{Im}U(\mathbf{x},t).$$

Take the potential to be time-independent. Show, with the aid of the divergence theorem, that

$$\frac{d}{dt}\int_{\mathbb{R}^3}\rho(\mathbf{x},t)\,dV = \frac{2}{\hbar}\int_{\mathbb{R}^3}\rho(\mathbf{x},t)\,\mathrm{Im}U(\mathbf{x})\,dV.$$

Assuming the wavefunction $\psi(\mathbf{x},0)$ is normalised to unity, show that if $\rho(\mathbf{x},t)$ is expanded about $t=0$ so that $\rho(\mathbf{x},t) = \rho_0(\mathbf{x}) + t\rho_1(\mathbf{x}) + \cdots$, then

$$\int_{\mathbb{R}^3}\rho(\mathbf{x},t)\,dV = 1 + \frac{2t}{\hbar}\int_{\mathbb{R}^3}\rho_0(\mathbf{x})\,\mathrm{Im}U(\mathbf{x})\,dV + \cdots\,.$$

As time increases, how does the quantity on the left of this equation behave if $\mathrm{Im}U(\mathbf{x}) < 0$?

17D Electromagnetism

Three sides of a closed rectangular circuit C are fixed and one is moving. The circuit lies in the plane $z = 0$ and the sides are $x = 0$, $y = 0$, $x = a(t)$, $y = b$, where $a(t)$ is a given function of time. A magnetic field $\mathbf{B} = (0, 0, \frac{\partial f}{\partial x})$ is applied, where $f(x,t)$ is a given function of x and t only. Find the magnetic flux Φ of $\mathbf{B}$ through the surface S bounded by C.

Find an electric field $\mathbf{E_0}$ that satisfies the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and then write down the most general solution $\mathbf{E}$ in terms of $\mathbf{E}_0$ and an undetermined scalar function independent of f.

Verify that

$$\oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = -\frac{d\Phi}{dt},$$

where $\mathbf{v}$ is the velocity of the relevant side of C. Interpret the left hand side of this equation.

If a unit current flows round C, what is the rate of work required to maintain the motion of the moving side of the rectangle? You should ignore any electromagnetic fields produced by the current.

18A Fluid Dynamics

A layer of incompressible fluid of density ρ and viscosity μ flows steadily down a plane inclined at an angle θ to the horizontal. The layer is of uniform thickness h measured perpendicular to the plane and the viscosity of the overlying air can be neglected. Using coordinates x parallel to the plane (in steepest downwards direction) and y normal to the plane, write down the equations of motion and the boundary conditions on the plane and on the free top surface. Determine the pressure and velocity fields and show that the volume flux down the plane is

$$\frac{\rho g h^3 \sin\theta}{3\mu}.$$

Consider now the case where a second layer of fluid, of uniform thickness αh, viscosity $\beta\mu$ and density ρ, flows steadily on top of the first layer. Explain why one of the appropriate boundary conditions between the two fluids is

$$\mu \frac{\partial}{\partial y} u(h_-) = \beta\mu \frac{\partial}{\partial y} u(h_+)\,,$$

where u is the component of velocity in the x direction and h_- and h_+ refer to just below and just above the boundary respectively. Determine the velocity field in each layer.

19C Numerical Analysis

Let

$$f'(0) \approx a_0 f(0) + a_1 f(1) + a_2 f(2) =: \lambda(f)$$

be a formula of numerical differentiation which is exact on polynomials of degree 2, and let

$$e(f) = f'(0) - \lambda(f)$$

be its error.

Find the values of the coefficients a_0, a_1, a_2.

Using the Peano kernel theorem, find the least constant c such that, for all functions $f \in C^3[0,2]$, we have

$$|e(f)| \leqslant c \, \|f'''\|_\infty \, .$$

20H Statistics

Suppose x_1 is a single observation from a distribution with density f over $[0,1]$. It is desired to test $H_0 : f(x) = 1$ against $H_1 : f(x) = 2x$.

Let $\delta : [0,1] \to \{0,1\}$ define a test by $\delta(x_1) = i \iff$ 'accept H_i'. Let $\alpha_i(\delta) = P(\delta(x_1) = 1 - i \mid H_i)$. State the Neyman-Pearson lemma using this notation.

Let δ be the best test of size 0.10. Find δ and $\alpha_1(\delta)$.

Consider now $\delta : [0,1] \to \{0,1,\star\}$ where $\delta(x_1) = \star$ means 'declare the test to be inconclusive'. Let $\gamma_i(\delta) = P(\delta(x) = \star \mid H_i)$. Given prior probabilities π_0 for H_0 and $\pi_1 = 1 - \pi_0$ for H_1, and some w_0, w_1, let

$$\text{cost}(\delta) = \pi_0\big(w_0\alpha_0(\delta) + \gamma_0(\delta)\big) + \pi_1\big(w_1\alpha_1(\delta) + \gamma_1(\delta)\big).$$

Let $\delta^*(x_1) = i \iff x_1 \in A_i$, where $A_0 = [0, 0.5)$, $A_\star = [0.5, 0.6)$, $A_1 = [0.6, 1]$. Prove that for each value of $\pi_0 \in (0,1)$ there exist w_0, w_1 (depending on π_0) such that $\text{cost}(\delta^*) = \min_\delta \text{cost}(\delta)$. [*Hint*: $w_0 = 1 + 2(0.6)(\pi_1/\pi_0)$.]

Hence prove that if δ is any test for which

$$\alpha_i(\delta) \leqslant \alpha_i(\delta^*), \quad i = 0, 1$$

then $\gamma_0(\delta) \geqslant \gamma_0(\delta^*)$ and $\gamma_1(\delta) \geqslant \gamma_1(\delta^*)$.

21H Optimization

Use the two phase method to find all optimal solutions to the problem

$$\begin{aligned} \text{maximize} \;\; & 2x_1 + 3x_2 + x_3 \\ \text{subject to} \;\; & x_1 + x_2 + x_3 \leqslant 40 \\ & 2x_1 + x_2 - x_3 \geqslant 10 \\ & -x_2 + x_3 \geqslant 10 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

Suppose that the values $(40, 10, 10)$ are perturbed to $(40, 10, 10) + (\epsilon_1, \epsilon_2, \epsilon_3)$. Find an expression for the change in the optimal value, which is valid for all sufficiently small values of $\epsilon_1, \epsilon_2, \epsilon_3$.

Suppose that $(\epsilon_1, \epsilon_2, \epsilon_3) = (\theta, -2\theta, 0)$. For what values of θ is your expression valid?

END OF PAPER