UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday, 4 June, 2013 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

What is the adjugate of an $n \times n$ matrix A? How is it related to A^{-1}? Suppose all the entries of A are integers. Show that all the entries of A^{-1} are integers if and only if $\det A = \pm 1$.

2D Complex Analysis or Complex Methods

Classify the singularities (in the finite complex plane) of the following functions:

(i) $\dfrac{1}{(\cosh z)^2}$;

(ii) $\dfrac{1}{\cos(1/z)}$;

(iii) $\dfrac{1}{\log z} \quad (-\pi < \arg z < \pi)$;

(iv) $\dfrac{z^{\frac{1}{2}} - 1}{\sin \pi z} \quad (-\pi < \arg z < \pi)$.

3F Geometry

Let l_1 and l_2 be ultraparallel geodesics in the hyperbolic plane. Prove that the l_i have a unique common perpendicular.

Suppose now l_1, l_2, l_3 are pairwise ultraparallel geodesics in the hyperbolic plane. Can the three common perpendiculars be pairwise disjoint? Must they be pairwise disjoint? Briefly justify your answers.

4A Variational Principles

(a) Define what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be convex. Assuming g'' exists, state an equivalent condition. Let $f(x) = x \log x$, defined on $x > 0$. Show that $f(x)$ is convex.

(b) Find the Legendre transform $f^*(p)$ of $f(x) = x \log x$. State the domain of $f^*(p)$. Without further calculation, explain why $(f^*)^* = f$ in this case.

5A Fluid Dynamics

A two-dimensional flow is given by

$$\mathbf{u} = (x, -y + t)\,.$$

Show that the flow is both irrotational and incompressible. Find a stream function $\psi(x, y)$ such that $\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$. Sketch the streamlines at $t = 0$.

Find the pathline of a fluid particle that passes through (x_0, y_0) at $t = 0$ in the form $y = f(x, x_0, y_0)$ and sketch the pathline for $x_0 = 1$, $y_0 = 1$.

6C Numerical Analysis

Determine the nodes x_1, x_2 of the two-point Gaussian quadrature

$$\int_0^1 f(x) w(x)\, dx \approx a_1 f(x_1) + a_2 f(x_2), \qquad w(x) = x,$$

and express the coefficients a_1, a_2 in terms of x_1, x_2. [You don't need to find numerical values of the coefficients.]

7H Statistics

Let $x_1, \ldots, x_n$ be independent and identically distributed observations from a distribution with probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda(x-\mu)}, & x \geqslant \mu, \\ 0, & x < \mu, \end{cases}$$

where λ and μ are unknown positive parameters. Let $\beta = \mu + 1/\lambda$. Find the maximum likelihood estimators $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\beta}$.

Determine for each of $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\beta}$ whether or not it has a positive bias.

8H Optimization

State sufficient conditions for p and q to be optimal mixed strategies for the row and column players in a zero-sum game with payoff matrix A and value v.

Rowena and Colin play a hide-and-seek game. Rowena hides in one of 3 locations, and then Colin searches them in some order. If he searches in order i, j, k then his search cost is c_i, $c_i + c_j$ or $c_i + c_j + c_k$, depending upon whether Rowena hides in i, j or k, respectively, and where c_1, c_2, c_3 are all positive. Rowena (Colin) wishes to maximize (minimize) the expected search cost.

Formulate the payoff matrix for this game.

Let $c = c_1+c_2+c_3$. Suppose that Colin starts his search in location i with probability c_i/c, and then, if he does not find Rowena, he searches the remaining two locations in random order. What bound does this strategy place on the value of the game?

Guess Rowena's optimal hiding strategy, show that it is optimal and find the value of the game.

SECTION II

9E Linear Algebra

If V_1 and V_2 are vector spaces, what is meant by $V_1 \oplus V_2$? If V_1 and V_2 are subspaces of a vector space V, what is meant by $V_1 + V_2$?

Stating clearly any theorems you use, show that if V_1 and V_2 are subspaces of a finite dimensional vector space V, then

$$\dim V_1 + \dim V_2 = \dim(V_1 \cap V_2) + \dim(V_1 + V_2).$$

Let $V_1, V_2 \subset \mathbb{R}^4$ be subspaces with bases

$$V_1 = \langle (3,2,4,-1), (1,2,1,-2), (-2,3,3,2) \rangle$$
$$V_2 = \langle (1,4,2,4), (-1,1,-1,-1), (3,1,2,0) \rangle.$$

Find a basis $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ for $V_1 \cap V_2$ such that the first component of $\mathbf{v}_1$ and the second component of $\mathbf{v}_2$ are both 0.

10G Groups, Rings and Modules

(i) Consider the group $G = GL_2(\mathbb{R})$ of all 2 by 2 matrices with entries in $\mathbb{R}$ and non-zero determinant. Let T be its subgroup consisting of all diagonal matrices, and N be the normaliser of T in G. Show that N is generated by T and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and determine the quotient group N/T.

(ii) Now let p be a prime number, and F be the field of integers modulo p. Consider the group $G = GL_2(F)$ as above but with entries in F, and define T and N similarly. Find the order of the group N.

11F Analysis II

Define what it means for a sequence of functions $k_n : A \to \mathbb{R}$, $n = 1, 2, \ldots$, to *converge uniformly* on an interval $A \subset \mathbb{R}$.

By considering the functions $k_n(x) = \frac{\sin(nx)}{\sqrt{n}}$, or otherwise, show that uniform convergence of a sequence of differentiable functions does not imply uniform convergence of their derivatives.

Now suppose $k_n(x)$ is continuously differentiable on A for each n, that $k_n(x_0)$ converges as $n \to \infty$ for some $x_0 \in A$, and moreover that the derivatives $k_n'(x)$ converge uniformly on A. Prove that $k_n(x)$ converges to a continuously differentiable function $k(x)$ on A, and that

$$k'(x) = \lim_{n\to\infty} k_n'(x).$$

Hence, or otherwise, prove that the function

$$\sum_{n=1}^{\infty} \frac{x^n \sin(nx)}{n^3+1}$$

is continuously differentiable on $(-1, 1)$.

12G Metric and Topological Spaces

Consider the sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$, a subset of $\mathbb{R}^3$, as a subspace of $\mathbb{R}^3$ with the Euclidean metric.

(i) Show that S^2 is compact and Hausdorff as a topological space.

(ii) Let $X = S^2/\sim$ be the quotient set with respect to the equivalence relation identifying the antipodes, i.e.

$$(x,y,z) \sim (x',y',z') \iff (x',y',z') = (x,y,z) \text{ or } (-x,-y,-z).$$

Show that X is compact and Hausdorff with respect to the quotient topology.

13E Complex Analysis or Complex Methods

Suppose $p(z)$ is a polynomial of even degree, all of whose roots satisfy $|z| < R$. Explain why there is a holomorphic (*i.e.* analytic) function $h(z)$ defined on the region $R < |z| < \infty$ which satisfies $h(z)^2 = p(z)$. We write $h(z) = \sqrt{p(z)}$.

By expanding in a Laurent series or otherwise, evaluate

$$\int_C \sqrt{z^4 - z}\, dz$$

where C is the circle of radius 2 with the anticlockwise orientation. (Your answer will be well-defined up to a factor of ± 1, depending on which square root you pick.)

14B Methods

(i) Let $f(x) = x$, $0 < x \leqslant \pi$. Obtain the Fourier sine series and sketch the odd and even periodic extensions of $f(x)$ over the interval $-2\pi \leqslant x \leqslant 2\pi$. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

(ii) Consider the eigenvalue problem

$$\mathcal{L}y = -\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = \lambda y, \qquad \lambda \in \mathbb{R}$$

with boundary conditions $y(0) = y(\pi) = 0$. Find the eigenvalues and corresponding eigenfunctions. Recast $\mathcal{L}$ in Sturm-Liouville form and give the orthogonality condition for the eigenfunctions. Using the Fourier sine series obtained in part (i), or otherwise, and assuming completeness of the eigenfunctions, find a series for y that satisfies

$$\mathcal{L}y = xe^{-x}$$

for the given boundary conditions.

15B Quantum Mechanics

A particle with momentum $\hat{p}$ moves in a one-dimensional real potential with Hamiltonian given by

$$\hat{H} = \frac{1}{2m}(\hat{p} + isA)(\hat{p} - isA), \quad -\infty < x < \infty$$

where A is a real function and $s \in \mathbb{R}^+$. Obtain the potential energy of the system. Find $\chi(x)$ such that $(\hat{p} - isA)\chi(x) = 0$. Now, putting $A = x^n$, for $n \in \mathbb{Z}^+$, show that $\chi(x)$ can be normalised only if n is odd. Letting $n = 1$, use the inequality

$$\int_{-\infty}^{\infty} \psi^*(x)\hat{H}\psi(x)dx \geqslant 0$$

to show that

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}$$

assuming that both $\langle \hat{p} \rangle$ and $\langle \hat{x} \rangle$ vanish.

16D Electromagnetism

Briefly explain the main assumptions leading to Drude's theory of conductivity. Show that these assumptions lead to the following equation for the average drift velocity $\langle \mathbf{v}(t) \rangle$ of the conducting electrons:

$$\frac{d\langle \mathbf{v} \rangle}{dt} = -\tau^{-1}\langle \mathbf{v} \rangle + (e/m)\mathbf{E}$$

where m and e are the mass and charge of each conducting electron, τ^{-1} is the probability that a given electron collides with an ion in unit time, and $\mathbf{E}$ is the applied electric field.

Given that $\langle \mathbf{v} \rangle = \mathbf{v}_0 e^{-i\omega t}$ and $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$, where $\mathbf{v}_0$ and $\mathbf{E}_0$ are independent of t, show that

$$\mathbf{J} = \sigma \mathbf{E}\,. \qquad (*)$$

Here, $\sigma = \sigma_s/(1 - i\omega\tau)$, $\sigma_s = ne^2\tau/m$ and n is the number of conducting electrons per unit volume.

Now let $\mathbf{v}_0 = \widetilde{\mathbf{v}}_0 e^{i\mathbf{k}\cdot\mathbf{x}}$ and $\mathbf{E}_0 = \widetilde{\mathbf{E}}_0 e^{i\mathbf{k}\cdot\mathbf{x}}$, where $\widetilde{\mathbf{v}}_0$ and $\widetilde{\mathbf{E}}_0$ are constant. Assuming that $(*)$ remains valid, use Maxwell's equations (taking the charge density to be everywhere zero but allowing for a non-zero current density) to show that

$$k^2 = \frac{\omega^2}{c^2}\epsilon_r$$

where the relative permittivity $\epsilon_r = 1 + i\sigma/(\omega\epsilon_0)$ and $k = |\mathbf{k}|$.

In the case $\omega\tau \gg 1$ and $\omega < \omega_p$, where $\omega_p^2 = \sigma_s/\tau\epsilon_0$, show that the wave decays exponentially with distance inside the conductor.

17A Fluid Dynamics

Starting from the Euler momentum equation, derive the form of Bernoulli's equation appropriate for an unsteady irrotational motion of an inviscid incompressible fluid.

Water of density ρ is driven through a horizontal tube of length L and internal radius a from a water-filled balloon attached to one end of the tube. Assume that the pressure exerted by the balloon is proportional to its current volume (in excess of atmospheric pressure). Also assume that water exits the tube at atmospheric pressure, and that gravity may be neglected. Show that the time for the balloon to empty does not depend on its initial volume. Find the maximum speed of water exiting the pipe.

18C Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A and explain how it can be used to solve the least squares problem of finding the vector $x^* \in \mathbb{R}^n$ which minimises $\|Ax - b\|$, where $b \in \mathbb{R}^m$, $m > n$, and the norm is the Euclidean one.

Define a Givens rotation $\Omega^{[p,q]}$ and show that it is an orthogonal matrix.

Using a Givens rotation, solve the least squares problem for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix},$$

giving both x^* and $\|Ax^* - b\|$.

19H Statistics

Consider the general linear model $Y = X\theta + \epsilon$ where X is a known $n \times p$ matrix, θ is an unknown $p \times 1$ vector of parameters, and ϵ is an $n \times 1$ vector of independent $N(0, \sigma^2)$ random variables with unknown variance σ^2. Assume the $p \times p$ matrix $X^T X$ is invertible. Let

$$\hat{\theta} = (X^T X)^{-1} X^T Y$$
$$\hat{\epsilon} = Y - X\hat{\theta}.$$

What are the distributions of $\hat{\theta}$ and $\hat{\epsilon}$? Show that $\hat{\theta}$ and $\hat{\epsilon}$ are uncorrelated.

Four apple trees stand in a 2×2 rectangular grid. The annual yield of the tree at coordinate (i, j) conforms to the model

$$y_{ij} = \alpha_i + \beta x_{ij} + \epsilon_{ij}, \quad i, j \in \{1, 2\},$$

where x_{ij} is the amount of fertilizer applied to tree (i, j), α_1, α_2 may differ because of varying soil across rows, and the ϵ_{ij} are $N(0, \sigma^2)$ random variables that are independent of one another and from year to year. The following two possible experiments are to be compared:

$$\mathrm{I}: \ (x_{ij}) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathrm{II}: \ (x_{ij}) = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}.$$

Represent these as general linear models, with $\theta = (\alpha_1, \alpha_2, \beta)$. Compare the variances of estimates of β under I and II.

With II the following yields are observed:

$$(y_{ij}) = \begin{pmatrix} 100 & 300 \\ 600 & 400 \end{pmatrix}.$$

Forecast the total yield that will be obtained next year if no fertilizer is used. What is the 95% predictive interval for this yield?

20H Markov Chains

A Markov chain has state space $\{a, b, c\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 3/5 & 2/5 \\ 3/4 & 0 & 1/4 \\ 2/3 & 1/3 & 0 \end{pmatrix},$$

where the rows 1,2,3 correspond to a, b, c, respectively. Show that this Markov chain is equivalent to a random walk on some graph with 6 edges.

Let $k(i, j)$ denote the mean first passage time from i to j.

(i) Find $k(a, a)$ and $k(a, b)$.

(ii) Given $X_0 = a$, find the expected number of steps until the walk first completes a step from b to c.

(iii) Suppose the distribution of X_0 is $(\pi_1, \pi_2, \pi_3) = (5, 4, 3)/12$. Let $\tau(a, b)$ be the least m such that $\{a, b\}$ appears as a subsequence of $\{X_0, X_1, \ldots, X_m\}$. By comparing the distributions of $\{X_0, X_1, \ldots, X_m\}$ and $\{X_m, \ldots, X_1, X_0\}$ show that $E\tau(a, b) = E\tau(b, a)$ and that

$$k(b, a) - k(a, b) = \sum_{i \in \{a,b,c\}} \pi_i \big[k(i, a) - k(i, b)\big].$$

END OF PAPER