UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2013

List of Courses

Paper 3, Section I
2F Analysis II

For each of the following sequences of functions on $[0,1]$, indexed by $n = 1, 2, \ldots$, determine whether or not the sequence has a pointwise limit, and if so, determine whether or not the convergence to the pointwise limit is uniform.

1. $f_n(x) = 1/(1 + n^2x^2)$
2. $g_n(x) = nx(1-x)^n$
3. $h_n(x) = \sqrt{n}x(1-x)^n$

Paper 4, Section I
3F Analysis II

State and prove the chain rule for differentiable mappings $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^k$.

Suppose now $F : \mathbb{R}^2 \to \mathbb{R}^2$ has image lying on the unit circle in $\mathbb{R}^2$. Prove that the determinant $\det(DF|_x)$ vanishes for every $x \in \mathbb{R}^2$.

Paper 2, Section I
3F Analysis II

Let $\mathcal{C}[a,b]$ denote the vector space of continuous real-valued functions on the interval $[a,b]$, and let $\mathcal{C}'[a,b]$ denote the subspace of continuously differentiable functions.

Show that $\|f\|_1 = \max|f| + \max|f'|$ defines a norm on $\mathcal{C}'[a,b]$. Show furthermore that the map $\Phi : f \mapsto f'((a+b)/2)$ takes the closed unit ball $\{\|f\|_1 \leqslant 1\} \subset \mathcal{C}'[a,b]$ to a bounded subset of $\mathbb{R}$.

If instead we had used the norm $\|f\|_0 = \max|f|$ restricted from $\mathcal{C}[a,b]$ to $\mathcal{C}'[a,b]$, would Φ take the closed unit ball $\{\|f\|_0 \leqslant 1\} \subset \mathcal{C}'[a,b]$ to a bounded subset of $\mathbb{R}$? Justify your answer.

Paper 1, Section II
11F Analysis II

Define what it means for a sequence of functions $k_n : A \to \mathbb{R}$, $n = 1, 2, \ldots$, to *converge uniformly* on an interval $A \subset \mathbb{R}$.

By considering the functions $k_n(x) = \frac{\sin(nx)}{\sqrt{n}}$, or otherwise, show that uniform convergence of a sequence of differentiable functions does not imply uniform convergence of their derivatives.

Now suppose $k_n(x)$ is continuously differentiable on A for each n, that $k_n(x_0)$ converges as $n \to \infty$ for some $x_0 \in A$, and moreover that the derivatives $k_n'(x)$ converge uniformly on A. Prove that $k_n(x)$ converges to a continuously differentiable function $k(x)$ on A, and that

$$k'(x) = \lim_{n\to\infty} k_n'(x).$$

Hence, or otherwise, prove that the function

$$\sum_{n=1}^{\infty} \frac{x^n \sin(nx)}{n^3 + 1}$$

is continuously differentiable on $(-1, 1)$.

Paper 4, Section II
12F Analysis II

State the contraction mapping theorem.

A metric space (X, d) is bounded if $\{d(x, y) \,|\, x, y \in X\}$ is a bounded subset of $\mathbb{R}$. Suppose (X, d) is complete and bounded. Let $\mathrm{Maps}(X, X)$ denote the set of continuous maps from X to itself. For $f, g \in \mathrm{Maps}(X, X)$, let

$$\delta(f, g) \;=\; \sup_{x\in X} d(f(x), g(x)).$$

Prove that $(\mathrm{Maps}(X, X), \delta)$ is a complete metric space. Is the subspace $\mathcal{C} \subset \mathrm{Maps}(X, X)$ of contraction mappings a complete subspace?

Let $\tau : \mathcal{C} \to X$ be the map which associates to any contraction its fixed point. Prove that τ is continuous.

Paper 3, Section II

12F Analysis II

For each of the following statements, provide a proof or justify a counterexample.

1. The norms $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ and $\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|$ on $\mathbb{R}^n$ are Lipschitz equivalent.
2. The norms $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ and $\|x\|_\infty = \max_i |x_i|$ on the vector space of sequences $(x_i)_{i \geqslant 1}$ with $\sum |x_i| < \infty$ are Lipschitz equivalent.
3. Given a linear function $\phi : V \to W$ between normed real vector spaces, there is some N for which $\|\phi(x)\| \leqslant N$ for every $x \in V$ with $\|x\| \leqslant 1$.
4. Given a linear function $\phi : V \to W$ between normed real vector spaces for which there is some N for which $\|\phi(x)\| \leqslant N$ for every $x \in V$ with $\|x\| \leqslant 1$, then ϕ is continuous.
5. The uniform norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ is complete on the vector space of continuous real-valued functions f on $\mathbb{R}$ for which $f(x) = 0$ for $|x|$ sufficiently large.
6. The uniform norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ is complete on the vector space of continuous real-valued functions f on $\mathbb{R}$ which are bounded.

Paper 2, Section II

12F Analysis II

Let $f : U \to \mathbb{R}$ be continuous on an open set $U \subset \mathbb{R}^2$. Suppose that on U the partial derivatives $D_1 f$, $D_2 f$, $D_1 D_2 f$ and $D_2 D_1 f$ exist and are continuous. Prove that $D_1 D_2 f \;=\; D_2 D_1 f$ on U.

If f is infinitely differentiable, and $m \in \mathbb{N}$, what is the maximum number of distinct m-th order partial derivatives that f may have on U ?

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^4 + y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = \begin{cases} \dfrac{xy(x^4 - y^4)}{x^4 + y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

For each of f and g, determine whether they are (i) differentiable, (ii) infinitely differentiable at the origin. Briefly justify your answers.

Paper 4, Section I

4E Complex Analysis

State Rouché's theorem. How many roots of the polynomial $z^8 + 3z^7 + 6z^2 + 1$ are contained in the annulus $1 < |z| < 2$?

Paper 3, Section II

13E Complex Analysis

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk, and let C be its boundary (the unit circle), with the anticlockwise orientation. Suppose $\phi : C \to \mathbb{C}$ is continuous. Stating clearly any theorems you use, show that

$$g_\phi(w) = \frac{1}{2\pi i} \int_C \frac{\phi(z)}{z - w} dz$$

is an analytic function of w for $w \in D$.

Now suppose ϕ is the restriction of a holomorphic function F defined on some annulus $1 - \epsilon < |z| < 1 + \epsilon$. Show that $g_\phi(w)$ is the restriction of a holomorphic function defined on the open disc $|w| < 1 + \epsilon$.

Let $f_\phi : [0, 2\pi] \to \mathbb{C}$ be defined by $f_\phi(\theta) = \phi(e^{i\theta})$. Express the coefficients in the power series expansion of g_ϕ centered at 0 in terms of f_ϕ.

Let $n \in \mathbb{Z}$. What is g_ϕ in the following cases?

1. $\phi(z) = z^n$.
2. $\phi(z) = \overline{z}^n$.
3. $\phi(z) = (\mathrm{Re}\ z)^2$.

Paper 1, Section I

2D Complex Analysis or Complex Methods

Classify the singularities (in the finite complex plane) of the following functions:

(i) $\dfrac{1}{(\cosh z)^2}$;

(ii) $\dfrac{1}{\cos(1/z)}$;

(iii) $\dfrac{1}{\log z} \quad (-\pi < \arg z < \pi)$;

(iv) $\dfrac{z^{\frac{1}{2}} - 1}{\sin \pi z} \quad (-\pi < \arg z < \pi)$.

Paper 1, Section II

13E Complex Analysis or Complex Methods

Suppose $p(z)$ is a polynomial of even degree, all of whose roots satisfy $|z| < R$. Explain why there is a holomorphic (*i.e.* analytic) function $h(z)$ defined on the region $R < |z| < \infty$ which satisfies $h(z)^2 = p(z)$. We write $h(z) = \sqrt{p(z)}$.

By expanding in a Laurent series or otherwise, evaluate

$$\int_C \sqrt{z^4 - z}\, dz$$

where C is the circle of radius 2 with the anticlockwise orientation. (Your answer will be well-defined up to a factor of ± 1, depending on which square root you pick.)

Paper 2, Section II

13D Complex Analysis or Complex Methods

Let

$$I = \oint_C \frac{e^{iz^2/\pi}}{1+e^{-2z}} dz\,,$$

where C is the rectangle with vertices at $\pm R$ and $\pm\ R + i\pi$, traversed anti-clockwise.

(i) Show that $I = \dfrac{\pi(1+i)}{\sqrt{2}}$.

(ii) Assuming that the contribution to I from the vertical sides of the rectangle is negligible in the limit $R \to \infty$, show that

$$\int_{-\infty}^{\infty} e^{ix^2/\pi} dx = \frac{\pi(1+i)}{\sqrt{2}}\,.$$

(iii) Justify briefly the assumption that the contribution to I from the vertical sides of the rectangle is negligible in the limit $R \to \infty$.

Paper 3, Section I

4D Complex Methods

Let $y(t) = 0$ for $t < 0$, and let $\lim_{t \to 0^+} y(t) = y_0$.

(i) Find the Laplace transforms of $H(t)$ and $tH(t)$, where $H(t)$ is the Heaviside step function.

(ii) Given that the Laplace transform of $y(t)$ is $\widehat{y}(s)$, find expressions for the Laplace transforms of $\dot{y}(t)$ and $y(t-1)$.

(iii) Use Laplace transforms to solve the equation

$$\dot{y}(t) - y(t-1) = H(t) - (t-1)H(t-1)$$

in the case $y_0 = 0$.

Paper 4, Section II

14D Complex Methods

Let C_1 and C_2 be the circles $x^2 + y^2 = 1$ and $5x^2 - 4x + 5y^2 = 0$, respectively, and let D be the (finite) region between the circles. Use the conformal mapping

$$w = \frac{z-2}{2z-1}$$

to solve the following problem:

$$\nabla^2 \phi = 0 \ \text{ in } D \text{ with } \phi = 1 \text{ on } C_1 \text{ and } \phi = 2 \text{ on } C_2.$$

Paper 2, Section I

6D Electromagnetism

Use Maxwell's equations to obtain the equation of continuity

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0\,.$$

Show that, for a body made from material of uniform conductivity σ, the charge density at any fixed internal point decays exponentially in time. If the body is finite and isolated, explain how this result can be consistent with overall charge conservation.

Paper 4, Section I

7D Electromagnetism

The infinite plane $z = 0$ is earthed and the infinite plane $z = d$ carries a charge of σ per unit area. Find the electrostatic potential between the planes.

Show that the electrostatic energy per unit area (of the planes $z =$ constant) between the planes can be written as either $\frac{1}{2}\sigma^2 d/\epsilon_0$ or $\frac{1}{2}\epsilon_0 V^2/d$, where V is the potential at $z = d$.

The distance between the planes is now increased by αd, where α is small. Show that the change in the energy per unit area is $\frac{1}{2}\sigma V\alpha$ if the upper plane ($z = d$) is electrically isolated, and is approximately $-\frac{1}{2}\sigma V\alpha$ if instead the potential on the upper plane is maintained at V. Explain briefly how this difference can be accounted for.

Paper 1, Section II

16D Electromagnetism

Briefly explain the main assumptions leading to Drude's theory of conductivity. Show that these assumptions lead to the following equation for the average drift velocity $\langle \mathbf{v}(t) \rangle$ of the conducting electrons:

$$\frac{d\langle \mathbf{v} \rangle}{dt} = -\tau^{-1}\langle \mathbf{v} \rangle + (e/m)\mathbf{E}$$

where m and e are the mass and charge of each conducting electron, τ^{-1} is the probability that a given electron collides with an ion in unit time, and $\mathbf{E}$ is the applied electric field.

Given that $\langle \mathbf{v} \rangle = \mathbf{v}_0 e^{-i\omega t}$ and $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$, where $\mathbf{v}_0$ and $\mathbf{E}_0$ are independent of t, show that

$$\mathbf{J} = \sigma \mathbf{E}\,. \qquad (*)$$

Here, $\sigma = \sigma_s/(1 - i\omega\tau)$, $\sigma_s = ne^2\tau/m$ and n is the number of conducting electrons per unit volume.

Now let $\mathbf{v}_0 = \widetilde{\mathbf{v}}_0 e^{i\mathbf{k}\cdot\mathbf{x}}$ and $\mathbf{E}_0 = \widetilde{\mathbf{E}}_0 e^{i\mathbf{k}\cdot\mathbf{x}}$, where $\widetilde{\mathbf{v}}_0$ and $\widetilde{\mathbf{E}}_0$ are constant. Assuming that $(*)$ remains valid, use Maxwell's equations (taking the charge density to be everywhere zero but allowing for a non-zero current density) to show that

$$k^2 = \frac{\omega^2}{c^2}\epsilon_r$$

where the relative permittivity $\epsilon_r = 1 + i\sigma/(\omega\epsilon_0)$ and $k = |\mathbf{k}|$.

In the case $\omega\tau \gg 1$ and $\omega < \omega_p$, where $\omega_p^2 = \sigma_s/\tau\epsilon_0$, show that the wave decays exponentially with distance inside the conductor.

Paper 3, Section II

17D Electromagnetism

Three sides of a closed rectangular circuit C are fixed and one is moving. The circuit lies in the plane $z = 0$ and the sides are $x = 0$, $y = 0$, $x = a(t)$, $y = b$, where $a(t)$ is a given function of time. A magnetic field $\mathbf{B} = (0, 0, \frac{\partial f}{\partial x})$ is applied, where $f(x, t)$ is a given function of x and t only. Find the magnetic flux Φ of $\mathbf{B}$ through the surface S bounded by C.

Find an electric field $\mathbf{E_0}$ that satisfies the Maxwell equation

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and then write down the most general solution $\mathbf{E}$ in terms of $\mathbf{E}_0$ and an undetermined scalar function independent of f.

Verify that

$$\oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = -\frac{d\Phi}{dt},$$

where $\mathbf{v}$ is the velocity of the relevant side of C. Interpret the left hand side of this equation.

If a unit current flows round C, what is the rate of work required to maintain the motion of the moving side of the rectangle? You should ignore any electromagnetic fields produced by the current.

Paper 2, Section II

18D Electromagnetism

Starting with the expression

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\,dV'}{|\mathbf{r}-\mathbf{r}'|}$$

for the magnetic vector potential at the point $\mathbf{r}$ due to a current distribution of density $\mathbf{J}(\mathbf{r})$, obtain the Biot-Savart law for the magnetic field due to a current I flowing in a simple loop C:

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\oint_C \frac{d\mathbf{r}'\times(\mathbf{r}'-\mathbf{r})}{|\mathbf{r}'-\mathbf{r}|^3} \qquad (\mathbf{r}\notin C).$$

Verify by direct differentiation that this satisfies $\boldsymbol{\nabla}\times\mathbf{B}=\mathbf{0}$. You may use without proof the identity $\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{v}) = \mathbf{a}(\boldsymbol{\nabla}\cdot\mathbf{v}) - (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{v}$, where $\mathbf{a}$ is a constant vector and $\mathbf{v}$ is a vector field.

Given that C is planar, and is described in cylindrical polar coordinates by $z=0$, $r=f(\theta)$, show that the magnetic field at the origin is

$$\widehat{\mathbf{z}}\,\frac{\mu_0 I}{4\pi}\oint\frac{d\theta}{f(\theta)}\,.$$

If C is the ellipse $r(1-e\cos\theta)=\ell$, find the magnetic field at the focus due to a current I.

Paper 1, Section I

5A Fluid Dynamics

A two-dimensional flow is given by

$$\mathbf{u} = (x, -y + t)\,.$$

Show that the flow is both irrotational and incompressible. Find a stream function $\psi(x, y)$ such that $\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$. Sketch the streamlines at $t = 0$.

Find the pathline of a fluid particle that passes through (x_0, y_0) at $t = 0$ in the form $y = f(x, x_0, y_0)$ and sketch the pathline for $x_0 = 1$, $y_0 = 1$.

Paper 2, Section I

7A Fluid Dynamics

An incompressible, inviscid fluid occupies the region beneath the free surface $y = \eta(x, t)$ and moves with a velocity field determined by the velocity potential $\phi(x, y, t)$. Gravity acts in the $-y$ direction. You may assume Bernoulli's integral of the equation of motion:

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + gy = F(t)\,.$$

Give the kinematic and dynamic boundary conditions that must be satisfied by ϕ on $y = \eta(x, t)$.

In the absence of waves, the fluid has constant uniform velocity U in the x direction. Derive the linearised form of the boundary conditions for small amplitude waves.

Assume that the free surface and velocity potential are of the form:

$$\begin{aligned} \eta &= a e^{i(kx - \omega t)} \\ \phi &= Ux + ib e^{ky} e^{i(kx - \omega t)} \end{aligned}$$

(where implicitly the real parts are taken). Show that

$$(\omega - kU)^2 = gk\,.$$

Paper 1, Section II
17A Fluid Dynamics

Starting from the Euler momentum equation, derive the form of Bernoulli's equation appropriate for an unsteady irrotational motion of an inviscid incompressible fluid.

Water of density ρ is driven through a horizontal tube of length L and internal radius a from a water-filled balloon attached to one end of the tube. Assume that the pressure exerted by the balloon is proportional to its current volume (in excess of atmospheric pressure). Also assume that water exits the tube at atmospheric pressure, and that gravity may be neglected. Show that the time for the balloon to empty does not depend on its initial volume. Find the maximum speed of water exiting the pipe.

Paper 4, Section II
18A Fluid Dynamics

The axisymmetric, irrotational flow generated by a solid sphere of radius a translating at velocity U in an inviscid, incompressible fluid is represented by a velocity potential $\phi(r,\theta)$. Assume the fluid is at rest far away from the sphere. Explain briefly why $\nabla^2\phi = 0$.

By trying a solution of the form $\phi(r,\theta) = f(r)\,g(\theta)$, show that

$$\phi = -\frac{Ua^3\cos\theta}{2r^2}$$

and write down the fluid velocity.

Show that the total kinetic energy of the fluid is $kMU^2/4$ where M is the mass of the sphere and k is the ratio of the density of the fluid to the density of the sphere.

A heavy sphere (i.e. $k<1$) is released from rest in an inviscid fluid. Determine its speed after it has fallen a distance h in terms of M, k, g and h.

Note, in spherical polars:

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right).$$

Paper 3, Section II

18A Fluid Dynamics

A layer of incompressible fluid of density ρ and viscosity μ flows steadily down a plane inclined at an angle θ to the horizontal. The layer is of uniform thickness h measured perpendicular to the plane and the viscosity of the overlying air can be neglected. Using coordinates x parallel to the plane (in steepest downwards direction) and y normal to the plane, write down the equations of motion and the boundary conditions on the plane and on the free top surface. Determine the pressure and velocity fields and show that the volume flux down the plane is

$$\frac{\rho g h^3 \sin\theta}{3\mu}.$$

Consider now the case where a second layer of fluid, of uniform thickness αh, viscosity $\beta\mu$ and density ρ, flows steadily on top of the first layer. Explain why one of the appropriate boundary conditions between the two fluids is

$$\mu \frac{\partial}{\partial y} u(h_-) = \beta\mu \frac{\partial}{\partial y} u(h_+)\,,$$

where u is the component of velocity in the x direction and h_- and h_+ refer to just below and just above the boundary respectively. Determine the velocity field in each layer.

Paper 1, Section I

3F Geometry

Let l_1 and l_2 be ultraparallel geodesics in the hyperbolic plane. Prove that the l_i have a unique common perpendicular.

Suppose now l_1, l_2, l_3 are pairwise ultraparallel geodesics in the hyperbolic plane. Can the three common perpendiculars be pairwise disjoint? Must they be pairwise disjoint? Briefly justify your answers.

Paper 3, Section I

5F Geometry

Let S be a surface with Riemannian metric having first fundamental form $du^2 + G(u,v)dv^2$. State a formula for the Gauss curvature K of S.

Suppose that S is flat, so K vanishes identically, and that $u = 0$ is a geodesic on S when parametrised by arc-length. Using the geodesic equations, or otherwise, prove that $G(u,v) \equiv 1$, i.e. S is locally isometric to a plane.

Paper 2, Section II

14F Geometry

Let A and B be disjoint circles in $\mathbb{C}$. Prove that there is a Möbius transformation which takes A and B to two concentric circles.

A collection of circles $X_i \subset \mathbb{C}$, $0 \leqslant i \leqslant n-1$, for which

1. X_i is tangent to A, B and X_{i+1}, where indices are *mod* n;
2. the circles are disjoint away from tangency points;

is called a *constellation* on (A,B). Prove that for any $n \geqslant 2$ there is some pair (A,B) and a constellation on (A,B) made up of precisely n circles. Draw a picture illustrating your answer.

Given a constellation on (A,B), prove that the tangency points $X_i \cap X_{i+1}$ for $0 \leqslant i \leqslant n-1$ all lie on a circle. Moreover, prove that if we take any other circle Y_0 tangent to A and B, and then construct Y_i for $i \geqslant 1$ inductively so that Y_i is tangent to A, B and Y_{i-1}, then we will have $Y_n = Y_0$, i.e. the chain of circles will again close up to form a constellation.

Paper 3, Section II

14F Geometry

Show that the set of all straight lines in $\mathbb{R}^2$ admits the structure of an abstract smooth surface S. Show that S is an open Möbius band (i.e. the Möbius band without its boundary circle), and deduce that S admits a Riemannian metric with vanishing Gauss curvature.

Show that there is no metric $d : S \times S \to \mathbb{R}_{\geqslant 0}$, in the sense of metric spaces, which

1. induces the locally Euclidean topology on S constructed above;
2. is invariant under the natural action on S of the group of translations of $\mathbb{R}^2$.

Show that the set of great circles on the two-dimensional sphere admits the structure of a smooth surface S'. Is S' homeomorphic to S ? Does S' admit a Riemannian metric with vanishing Gauss curvature? Briefly justify your answers.

Paper 4, Section II

15F Geometry

Let η be a smooth curve in the xz-plane $\eta(s) = (f(s), 0, g(s))$, with $f(s) > 0$ for every $s \in \mathbb{R}$ and $f'(s)^2 + g'(s)^2 = 1$. Let S be the surface obtained by rotating η around the z-axis. Find the first fundamental form of S.

State the equations for a curve $\gamma : (a, b) \to S$ parametrised by arc-length to be a geodesic.

A parallel on S is the closed circle swept out by rotating a single point of η. Prove that for every $n \in \mathbb{Z}_{>0}$ there is some η for which exactly n parallels are geodesics. Sketch possible such surfaces S in the cases $n = 1$ and $n = 2$.

If *every* parallel is a geodesic, what can you deduce about S ? Briefly justify your answer.

Paper 3, Section I

1G Groups, Rings and Modules

Define the notion of a free module over a ring. When R is a PID, show that every ideal of R is free as an R-module.

Paper 4, Section I

2G Groups, Rings and Modules

Let p be a prime number, and G be a non-trivial finite group whose order is a power of p. Show that the size of every conjugacy class in G is a power of p. Deduce that the centre Z of G has order at least p.

Paper 2, Section I

2G Groups, Rings and Modules

Show that every Euclidean domain is a PID. Define the notion of a Noetherian ring, and show that $\mathbb{Z}[i]$ is Noetherian by using the fact that it is a Euclidean domain.

Paper 1, Section II

10G Groups, Rings and Modules

(i) Consider the group $G = GL_2(\mathbb{R})$ of all 2 by 2 matrices with entries in $\mathbb{R}$ and non-zero determinant. Let T be its subgroup consisting of all diagonal matrices, and N be the normaliser of T in G. Show that N is generated by T and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and determine the quotient group N/T.

(ii) Now let p be a prime number, and F be the field of integers modulo p. Consider the group $G = GL_2(F)$ as above but with entries in F, and define T and N similarly. Find the order of the group N.

Paper 4, Section II
11G Groups, Rings and Modules

Let R be an integral domain, and M be a finitely generated R-module.

(i) Let S be a finite subset of M which generates M as an R-module. Let T be a maximal linearly independent subset of S, and let N be the R-submodule of M generated by T. Show that there exists a non-zero $r \in R$ such that $rx \in N$ for every $x \in M$.

(ii) Now assume M is *torsion-free*, i.e. $rx = 0$ for $r \in R$ and $x \in M$ implies $r = 0$ or $x = 0$. By considering the map $M \to N$ mapping x to rx for r as in (i), show that every torsion-free finitely generated R-module is isomorphic to an R-submodule of a finitely generated free R-module.

Paper 3, Section II
11G Groups, Rings and Modules

Let $R = \mathbb{C}[X, Y]$ be the polynomial ring in two variables over the complex numbers, and consider the principal ideal $I = (X^3 - Y^2)$ of R.

(i) Using the fact that R is a UFD, show that I is a prime ideal of R. [*Hint: Elements in* $\mathbb{C}[X, Y]$ *are polynomials in* Y *with coefficients in* $\mathbb{C}[X]$.]

(ii) Show that I is not a maximal ideal of R, and that it is contained in infinitely many distinct proper ideals in R.

Paper 2, Section II
11G Groups, Rings and Modules

(i) State the structure theorem for finitely generated modules over Euclidean domains.

(ii) Let $\mathbb{C}[X]$ be the polynomial ring over the complex numbers. Let M be a $\mathbb{C}[X]$-module which is 4-dimensional as a $\mathbb{C}$-vector space and such that $(X-2)^4 \cdot x = 0$ for all $x \in M$. Find all possible forms we obtain when we write $M \cong \bigoplus_{i=1}^{m} \mathbb{C}[X]/(P_i^{n_i})$ for irreducible $P_i \in \mathbb{C}[X]$ and $n_i \geqslant 1$.

(iii) Consider the quotient ring $M = \mathbb{C}[X]/(X^3 + X)$ as a $\mathbb{C}[X]$-module. Show that M is isomorphic as a $\mathbb{C}[X]$-module to the direct sum of three copies of $\mathbb{C}$. Give the isomorphism and its inverse explicitly.

Paper 4, Section I

1E Linear Algebra

What is a quadratic form on a finite dimensional real vector space V? What does it mean for two quadratic forms to be isomorphic (*i.e.* congruent)? State Sylvester's law of inertia and explain the definition of the quantities which appear in it. Find the signature of the quadratic form on $\mathbb{R}^3$ given by $q(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$, where

$$A = \begin{pmatrix} -2 & 1 & 6 \\ 1 & -1 & -3 \\ 6 & -3 & 1 \end{pmatrix}.$$

Paper 2, Section I

1E Linear Algebra

If A is an $n \times n$ invertible Hermitian matrix, let

$$U_A = \{U \in M_{n\times n}(\mathbb{C}) \,|\, \overline{U}^T AU = A\}.$$

Show that U_A with the operation of matrix multiplication is a group, and that $\det U$ has norm 1 for any $U \in U_A$. What is the relation between U_A and the complex Hermitian form defined by A?

If $A = I_n$ is the $n \times n$ identity matrix, show that any element of U_A is diagonalizable.

Paper 1, Section I

1E Linear Algebra

What is the adjugate of an $n \times n$ matrix A? How is it related to A^{-1}? Suppose all the entries of A are integers. Show that all the entries of A^{-1} are integers if and only if $\det A = \pm 1$.

Paper 1, Section II

9E Linear Algebra

If V_1 and V_2 are vector spaces, what is meant by $V_1 \oplus V_2$? If V_1 and V_2 are subspaces of a vector space V, what is meant by $V_1 + V_2$?

Stating clearly any theorems you use, show that if V_1 and V_2 are subspaces of a finite dimensional vector space V, then

$$\dim V_1 + \dim V_2 = \dim(V_1 \cap V_2) + \dim(V_1 + V_2).$$

Let $V_1, V_2 \subset \mathbb{R}^4$ be subspaces with bases

$$\begin{aligned} V_1 &= \langle (3,2,4,-1), (1,2,1,-2), (-2,3,3,2) \rangle \\ V_2 &= \langle (1,4,2,4), (-1,1,-1,-1), (3,1,2,0) \rangle. \end{aligned}$$

Find a basis $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ for $V_1 \cap V_2$ such that the first component of $\mathbf{v}_1$ and the second component of $\mathbf{v}_2$ are both 0.

Paper 4, Section II

10E Linear Algebra

What does it mean for an $n \times n$ matrix to be in Jordan form? Show that if $A \in M_{n\times n}(\mathbb{C})$ is in Jordan form, there is a sequence (A_m) of diagonalizable $n \times n$ matrices which converges to A, in the sense that the (ij)th component of A_m converges to the (ij)th component of A for all i and j. [*Hint: A matrix with distinct eigenvalues is diagonalizable.*] Deduce that the same statement holds for all $A \in M_{n\times n}(\mathbb{C})$.

Let $V = M_{2\times 2}(\mathbb{C})$. Given $A \in V$, define a linear map $T_A : V \to V$ by $T_A(B) = AB + BA$. Express the characteristic polynomial of T_A in terms of the trace and determinant of A. [*Hint: First consider the case where A is diagonalizable.*]

Paper 3, Section II

10E Linear Algebra

Let V and W be finite dimensional real vector spaces and let $T : V \to W$ be a linear map. Define the dual space V^* and the dual map T^*. Show that there is an isomorphism $\iota : V \to (V^*)^*$ which is canonical, in the sense that $\iota \circ S = (S^*)^* \circ \iota$ for any automorphism S of V.

Now let W be an inner product space. Use the inner product to show that there is an injective map from $\operatorname{im} T$ to $\operatorname{im} T^*$. Deduce that the row rank of a matrix is equal to its column rank.

Paper 2, Section II

10E Linear Algebra

Define what it means for a set of vectors in a vector space V to be linearly dependent. Prove from the definition that any set of $n+1$ vectors in $\mathbb{R}^n$ is linearly dependent.

Using this or otherwise, prove that if V has a finite basis consisting of n elements, then any basis of V has exactly n elements.

Let V be the vector space of bounded continuous functions on $\mathbb{R}$. Show that V is infinite dimensional.

Paper 4, Section I

9H Markov Chains

Suppose P is the transition matrix of an irreducible recurrent Markov chain with state space I. Show that if x is an invariant measure and $x_k > 0$ for some $k \in I$, then $x_j > 0$ for all $j \in I$.

Let

$$\gamma_j^k = p_{kj} + \sum_{t=1}^{\infty} \sum_{i_1 \neq k, \ldots, i_t \neq k} p_{ki_t} p_{i_t i_{t-1}} \cdots p_{i_1 j}.$$

Give a meaning to γ_j^k and explain why $\gamma_k^k = 1$.

Suppose x is an invariant measure with $x_k = 1$. Prove that $x_j \geqslant \gamma_j^k$ for all j.

Paper 3, Section I

9H Markov Chains

Prove that if a distribution π is in detailed balance with a transition matrix P then it is an invariant distribution for P.

Consider the following model with 2 urns. At each time, $t = 0, 1, \ldots$ one of the following happens:

- with probability β a ball is chosen at random and moved to the other urn (but nothing happens if both urns are empty);
- with probability γ a ball is chosen at random and removed (but nothing happens if both urns are empty);
- with probability α a new ball is added to a randomly chosen urn,

where $\alpha + \beta + \gamma = 1$ and $\alpha < \gamma$. State (i, j) denotes that urns 1, 2 contain i and j balls respectively. Prove that there is an invariant measure

$$\lambda_{i,j} = \frac{(i+j)!}{i!j!} (\alpha/2\gamma)^{i+j}.$$

Find the proportion of time for which there are n balls in the system.

Paper 1, Section II

20H Markov Chains

A Markov chain has state space $\{a, b, c\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 3/5 & 2/5 \\ 3/4 & 0 & 1/4 \\ 2/3 & 1/3 & 0 \end{pmatrix},$$

where the rows 1,2,3 correspond to a, b, c, respectively. Show that this Markov chain is equivalent to a random walk on some graph with 6 edges.

Let $k(i, j)$ denote the mean first passage time from i to j.

(i) Find $k(a, a)$ and $k(a, b)$.

(ii) Given $X_0 = a$, find the expected number of steps until the walk first completes a step from b to c.

(iii) Suppose the distribution of X_0 is $(\pi_1, \pi_2, \pi_3) = (5, 4, 3)/12$. Let $\tau(a, b)$ be the least m such that $\{a, b\}$ appears as a subsequence of $\{X_0, X_1, \ldots, X_m\}$. By comparing the distributions of $\{X_0, X_1, \ldots, X_m\}$ and $\{X_m, \ldots, X_1, X_0\}$ show that $E\tau(a, b) = E\tau(b, a)$ and that

$$k(b, a) - k(a, b) = \sum_{i \in \{a,b,c\}} \pi_i \big[k(i, a) - k(i, b)\big].$$

Paper 2, Section II

20H Markov Chains

(i) Suppose $(X_n)_{n\geqslant 0}$ is an irreducible Markov chain and $f_{ij} = P(X_n = j \text{ for some } n \geqslant 1 \mid X_0 = i)$. Prove that $f_{ii} \geqslant f_{ij} f_{ji}$ and that

$$\sum_{n=0}^{\infty} P_i(X_n = i) = \sum_{n=1}^{\infty} f_{ii}^{n-1}.$$

(ii) Let $(X_n)_{n\geqslant 0}$ be a symmetric random walk on the $\mathbb{Z}^2$ lattice. Prove that $(X_n)_{n\geqslant 0}$ is recurrent. You may assume, for $n \geqslant 1$,

$$1/2 < 2^{-2n}\sqrt{n}\binom{2n}{n} < 1.$$

(iii) A princess and monster perform independent random walks on the $\mathbb{Z}^2$ lattice. The trajectory of the princess is the symmetric random walk $(X_n)_{n\geqslant 0}$. The monster's trajectory, denoted $(Z_n)_{n\geqslant 0}$, is a sleepy version of an independent symmetric random walk $(Y_n)_{n\geqslant 0}$. Specifically, given an infinite sequence of integers $0 = n_0 < n_1 < \cdots$, the monster sleeps between these times, so $Z_{n_i+1} = \cdots = Z_{n_{i+1}} = Y_{i+1}$. Initially, $X_0 = (100, 0)$ and $Z_0 = Y_0 = (0, 100)$. The princess is captured if and only if at some future time she and the monster are simultaneously at $(0, 0)$.

Compare the capture probabilities for an active monster, who takes $n_{i+1} = n_i + 1$ for all i, and a sleepy monster, who takes n_i spaced sufficiently widely so that

$$P\Big(X_k = (0,0) \text{ for some } k \in \{n_i + 1, \ldots, n_{i+1}\}\Big) > 1/2.$$

Paper 2, Section I

5B Methods

Consider the equation

$$xu_x + (x+y)u_y = 1$$

subject to the Cauchy data $u(1,y) = y$. Using the method of characteristics, obtain a solution to this equation.

Paper 4, Section I

5C Methods

Show that the general solution of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$$

can be written in the form

$$y(x,t) = f(ct - x) + g(ct + x)\,.$$

For the boundary conditions

$$y(0,t) = y(L,t) = 0, \qquad t > 0\,,$$

find the relation between f and g and show that they are $2L$-periodic. Hence show that

$$E(t) = \frac{1}{2}\int_0^L \left(\frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 \right) dx$$

is independent of t.

Paper 3, Section I

7C Methods

The solution to the Dirichlet problem on the half-space $D = \{\mathbf{x} = (x, y, z) : z > 0\}$:

$$\nabla^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in D\,, \qquad u(\mathbf{x}) \to 0 \quad \text{as} \ |\mathbf{x}| \to \infty, \qquad u(x, y, 0) = h(x, y),$$

is given by the formula

$$u(\mathbf{x}_0) = u(x_0, y_0, z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x, y)\frac{\partial}{\partial n}G(\mathbf{x}, \mathbf{x}_0)\,dx\,dy\,,$$

where n is the outward normal to ∂D.

State the boundary conditions on G and explain how G is related to G_3, where

$$G_3(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}_0|}$$

is the fundamental solution to the Laplace equation in three dimensions.

Using the method of images find an explicit expression for the function $\frac{\partial}{\partial n}G(\mathbf{x}, \mathbf{x}_0)$ in the formula.

Paper 1, Section II

14B Methods

(i) Let $f(x) = x$, $0 < x \leqslant \pi$. Obtain the Fourier sine series and sketch the odd and even periodic extensions of $f(x)$ over the interval $-2\pi \leqslant x \leqslant 2\pi$. Deduce that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

(ii) Consider the eigenvalue problem

$$\mathcal{L}y = -\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = \lambda y, \qquad \lambda \in \mathbb{R}$$

with boundary conditions $y(0) = y(\pi) = 0$. Find the eigenvalues and corresponding eigenfunctions. Recast $\mathcal{L}$ in Sturm-Liouville form and give the orthogonality condition for the eigenfunctions. Using the Fourier sine series obtained in part (i), or otherwise, and assuming completeness of the eigenfunctions, find a series for y that satisfies

$$\mathcal{L}y = xe^{-x}$$

for the given boundary conditions.

Paper 3, Section II

15C Methods

The Laplace equation in plane polar coordinates has the form

$$\nabla^2\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0\,.$$

Using separation of variables, derive the general solution to the equation that is single-valued in the domain $1 < r < 2$.

For

$$f(\theta) = \sum_{n=1}^{\infty} A_n \sin n\theta\,,$$

solve the Laplace equation in the annulus with the boundary conditions:

$$\nabla^2\phi = 0, \quad 1 < r < 2, \qquad \phi(r,\theta) = \begin{cases} f(\theta), & r = 1 \\ f(\theta) + 1, & r = 2. \end{cases}$$

Paper 2, Section II

16B Methods

The steady-state temperature distribution $u(x)$ in a uniform rod of finite length satisfies the boundary value problem

$$-D\frac{d^2}{dx^2}u(x) = f(x)\,, \qquad 0 < x < l$$

$$u(0) = 0\,, \qquad u(l) = 0$$

where $D > 0$ is the (constant) diffusion coefficient. Determine the Green's function $G(x,\xi)$ for this problem. Now replace the above homogeneous boundary conditions with the inhomogeneous boundary conditions $u(0) = \alpha,\quad u(l) = \beta$ and give a solution to the new boundary value problem. Hence, obtain the steady-state solution for the following problem with the specified boundary conditions:

$$-D\frac{\partial^2}{\partial x^2}u(x,t) + \frac{\partial}{\partial t}u(x,t) = x\,, \quad 0 < x < 1\,,$$

$$u(0,t) = 1/D\,, \qquad u(1,t) = 2/D\,, \qquad t > 0\,.$$

[You may assume that a steady-state solution exists.]

Paper 4, Section II

17C Methods

Find the inverse Fourier transform $G(x)$ of the function

$$g(k) = e^{-a|k|}, \qquad a > 0, \qquad -\infty < k < \infty\,.$$

Assuming that appropriate Fourier transforms exist, determine the solution $\psi(x, y)$ of

$$\nabla^2 \psi = 0, \qquad -\infty < x < \infty, \qquad 0 < y < 1,$$

with the following boundary conditions

$$\psi(x, 0) = \delta(x), \qquad \psi(x, 1) = \frac{1}{\pi}\frac{1}{x^2 + 1}\,.$$

Here $\delta(x)$ is the Dirac delta-function.

Paper 3, Section I

3G Metric and Topological Spaces

Let X be a metric space with the metric $d : X \times X \to \mathbb{R}$.

(i) Show that if X is compact as a topological space, then X is complete.

(ii) Show that the completeness of X is not a topological property, i.e. give an example of two metrics d, d' on a set X, such that the associated topologies are the same, but (X, d) is complete and (X, d') is not.

Paper 2, Section I

4G Metric and Topological Spaces

Let X be a topological space. Prove or disprove the following statements.

(i) If X is discrete, then X is compact if and only if it is a finite set.

(ii) If Y is a subspace of X and X, Y are both compact, then Y is closed in X.

Paper 1, Section II

12G Metric and Topological Spaces

Consider the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, a subset of $\mathbb{R}^3$, as a subspace of $\mathbb{R}^3$ with the Euclidean metric.

(i) Show that S^2 is compact and Hausdorff as a topological space.

(ii) Let $X = S^2/\sim$ be the quotient set with respect to the equivalence relation identifying the antipodes, i.e.

$$(x, y, z) \sim (x', y', z') \iff (x', y', z') = (x, y, z) \text{ or } (-x, -y, -z).$$

Show that X is compact and Hausdorff with respect to the quotient topology.

Paper 4, Section II

13G Metric and Topological Spaces

Let X be a topological space. A *connected component* of X means an equivalence class with respect to the equivalence relation on X defined as:

$$x \sim y \iff x, y \text{ belong to some connected subspace of } X.$$

(i) Show that every connected component is a connected and closed subset of X.

(ii) If X, Y are topological spaces and $X \times Y$ is the product space, show that every connected component of $X \times Y$ is a direct product of connected components of X and Y.

Paper 1, Section I

6C Numerical Analysis

Determine the nodes x_1, x_2 of the two-point Gaussian quadrature

$$\int_0^1 f(x)w(x)\,dx \approx a_1 f(x_1) + a_2 f(x_2), \qquad w(x) = x,$$

and express the coefficients a_1, a_2 in terms of x_1, x_2. [You don't need to find numerical values of the coefficients.]

Paper 4, Section I

8C Numerical Analysis

For a continuous function f, and $k+1$ distinct points $\{x_0, x_1, \ldots, x_k\}$, define the divided difference $f[x_0, \ldots, x_k]$ of order k.

Given $n+1$ points $\{x_0, x_1, \ldots, x_n\}$, let $p_n \in \mathbb{P}_n$ be the polynomial of degree n that interpolates f at these points. Prove that p_n can be written in the Newton form

$$p_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i)\,.$$

Paper 1, Section II

18C Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A and explain how it can be used to solve the least squares problem of finding the vector $x^* \in \mathbb{R}^n$ which minimises $\|Ax - b\|$, where $b \in \mathbb{R}^m$, $m > n$, and the norm is the Euclidean one.

Define a Givens rotation $\Omega^{[p,q]}$ and show that it is an orthogonal matrix.

Using a Givens rotation, solve the least squares problem for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix},$$

giving both x^* and $\|Ax^* - b\|$.

Paper 3, Section II

19C Numerical Analysis

Let

$$f'(0) \approx a_0 f(0) + a_1 f(1) + a_2 f(2) =: \lambda(f)$$

be a formula of numerical differentiation which is exact on polynomials of degree 2, and let

$$e(f) = f'(0) - \lambda(f)$$

be its error.

Find the values of the coefficients a_0, a_1, a_2.

Using the Peano kernel theorem, find the least constant c such that, for all functions $f \in C^3[0,2]$, we have

$$|e(f)| \leqslant c\,\|f'''\|_\infty\,.$$

Paper 2, Section II

19C Numerical Analysis

Explain briefly what is meant by the convergence of a numerical method for solving the ordinary differential equation

$$y'(t) = f(t,y), \qquad t \in [0,T], \qquad y(0) = y_0\,.$$

Prove from first principles that if the function f is sufficiently smooth and satisfies the Lipschitz condition

$$|f(t,x) - f(t,y)| \leqslant L\,|x-y|, \qquad x,y \in \mathbb{R}, \qquad t \in [0,T],$$

for some $L > 0$, then the backward Euler method

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})\,,$$

converges and find the order of convergence.

Find the linear stability domain of the backward Euler method.

Paper 1, Section I

8H Optimization

State sufficient conditions for p and q to be optimal mixed strategies for the row and column players in a zero-sum game with payoff matrix A and value v.

Rowena and Colin play a hide-and-seek game. Rowena hides in one of 3 locations, and then Colin searches them in some order. If he searches in order i, j, k then his search cost is c_i, $c_i + c_j$ or $c_i + c_j + c_k$, depending upon whether Rowena hides in i, j or k, respectively, and where c_1, c_2, c_3 are all positive. Rowena (Colin) wishes to maximize (minimize) the expected search cost.

Formulate the payoff matrix for this game.

Let $c = c_1 + c_2 + c_3$. Suppose that Colin starts his search in location i with probability c_i/c, and then, if he does not find Rowena, he searches the remaining two locations in random order. What bound does this strategy place on the value of the game?

Guess Rowena's optimal hiding strategy, show that it is optimal and find the value of the game.

Paper 2, Section I

9H Optimization

Given a network with a source A, a sink B, and capacities on directed arcs, define what is meant by a minimum cut.

The m streets and n intersections of a town are represented by sets of edges E and vertices V of a connected graph. A city planner wishes to make all streets one-way while ensuring it possible to drive away from each intersection along at least k different streets.

Use a theorem about min-cut and max-flow to prove that the city planner can achieve his goal provided that the following is true:

$$d(U) \geqslant k|U| \text{ for all } U \subseteq V,$$

where $|U|$ is the size of U and $d(U)$ is the number edges with at least one end in U. How could the planner find street directions that achieve his goal?

[*Hint: Consider a network having nodes A, B, nodes $a_1, \ldots, a_m$ for the streets and nodes $b_1, \ldots, b_n$ for the intersections. There are directed arcs from A to each a_i, and from each b_i to B. From each a_i there are two further arcs, directed towards b_j and $b_{j'}$ that correspond to endpoints of street i.*]

Paper 4, Section II

20H Optimization

Given real numbers a and b, consider the problem P of minimizing

$$f(x) = ax_{11} + 2x_{12} + 3x_{13} + bx_{21} + 4x_{22} + x_{23}$$

subject to $x_{ij} \geqslant 0$ and

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= 5 \\ x_{21} + x_{22} + x_{23} &= 5 \\ x_{11} + x_{21} &= 3 \\ x_{12} + x_{22} &= 3 \\ x_{13} + x_{23} &= 4. \end{aligned}$$

List all the basic feasible solutions, writing them as 2×3 matrices (x_{ij}).

Let $f(x) = \sum_{ij} c_{ij} x_{ij}$. Suppose there exist λ_i, μ_j such that

$$\lambda_i + \mu_j \leqslant c_{ij} \text{ for all } i \in \{1, 2\},\ j \in \{1, 2, 3\}\,.$$

Prove that if x and x' are both feasible for P and $\lambda_i + \mu_j = c_{ij}$ whenever $x_{ij} > 0$, then $f(x) \leqslant f(x')$.

Let x^* be the initial feasible solution that is obtained by formulating P as a transportation problem and using a greedy method that starts in the upper left of the matrix (x_{ij}). Show that if $a + 2 \leqslant b$ then x^* minimizes f.

For what values of a and b is one step of the transportation algorithm sufficient to pivot from x^* to a solution that *maximizes* f?

Paper 3, Section II

21H Optimization

Use the two phase method to find all optimal solutions to the problem

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 3x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leqslant 40 \\ & 2x_1 + x_2 - x_3 \geqslant 10 \\ & -x_2 + x_3 \geqslant 10 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

Suppose that the values $(40, 10, 10)$ are perturbed to $(40, 10, 10) + (\epsilon_1, \epsilon_2, \epsilon_3)$. Find an expression for the change in the optimal value, which is valid for all sufficiently small values of $\epsilon_1, \epsilon_2, \epsilon_3$.

Suppose that $(\epsilon_1, \epsilon_2, \epsilon_3) = (\theta, -2\theta, 0)$. For what values of θ is your expression valid?

Paper 4, Section I

6B Quantum Mechanics

The components of the three-dimensional angular momentum operator $\hat{\mathbf{L}}$ are defined as follows:

$$\hat{L}_x = -i\hbar\Big(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\Big) \quad \hat{L}_y = -i\hbar\Big(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\Big) \quad \hat{L}_z = -i\hbar\Big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\Big)\,.$$

Given that the wavefunction

$$\psi = (f(x) + iy)z$$

is an eigenfunction of $\hat{L}_z$, find all possible values of $f(x)$ and the corresponding eigenvalues of ψ. Letting $f(x) = x$, show that ψ is an eigenfunction of $\hat{\mathbf{L}}^2$ and calculate the corresponding eigenvalue.

Paper 3, Section I

8B Quantum Mechanics

If α, β and γ are linear operators, establish the identity

$$[\alpha\beta, \gamma] = \alpha[\beta, \gamma] + [\alpha, \gamma]\beta.$$

In what follows, the operators x and p are Hermitian and represent position and momentum of a quantum mechanical particle in one-dimension. Show that

$$[x^n, p] = i\hbar n x^{n-1}$$

and

$$[x, p^m] = i\hbar m p^{m-1}$$

where $m, n \in \mathbb{Z}^+$. Assuming $[x^n, p^m] \neq 0$, show that the operators x^n and p^m are Hermitian but their product is not. Determine whether $x^n p^m + p^m x^n$ is Hermitian.

Paper 1, Section II
15B Quantum Mechanics

A particle with momentum $\hat{p}$ moves in a one-dimensional real potential with Hamiltonian given by

$$\hat{H} = \frac{1}{2m}(\hat{p} + isA)(\hat{p} - isA), \quad -\infty < x < \infty$$

where A is a real function and $s \in \mathbb{R}^+$. Obtain the potential energy of the system. Find $\chi(x)$ such that $(\hat{p} - isA)\chi(x) = 0$. Now, putting $A = x^n$, for $n \in \mathbb{Z}^+$, show that $\chi(x)$ can be normalised only if n is odd. Letting $n = 1$, use the inequality

$$\int_{-\infty}^{\infty} \psi^*(x)\hat{H}\psi(x)dx \geqslant 0$$

to show that

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}$$

assuming that both $\langle \hat{p} \rangle$ and $\langle \hat{x} \rangle$ vanish.

Paper 3, Section II
16B Quantum Mechanics

Obtain, with the aid of the time-dependent Schrödinger equation, the conservation equation

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot \mathbf{j}(\mathbf{x}, t) = 0$$

where $\rho(\mathbf{x}, t)$ is the probability density and $\mathbf{j}(\mathbf{x}, t)$ is the probability current. What have you assumed about the potential energy of the system?

Show that if the potential $U(\mathbf{x}, t)$ is complex the conservation equation becomes

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot \mathbf{j}(\mathbf{x}, t) = \frac{2}{\hbar}\rho(\mathbf{x}, t)\,\mathrm{Im}U(\mathbf{x}, t).$$

Take the potential to be time-independent. Show, with the aid of the divergence theorem, that

$$\frac{d}{dt}\int_{\mathbb{R}^3} \rho(\mathbf{x}, t)\,dV = \frac{2}{\hbar}\int_{\mathbb{R}^3} \rho(\mathbf{x}, t)\,\mathrm{Im}U(\mathbf{x})\,dV.$$

Assuming the wavefunction $\psi(\mathbf{x}, 0)$ is normalised to unity, show that if $\rho(\mathbf{x}, t)$ is expanded about $t = 0$ so that $\rho(\mathbf{x}, t) = \rho_0(\mathbf{x}) + t\rho_1(\mathbf{x}) + \cdots$, then

$$\int_{\mathbb{R}^3} \rho(\mathbf{x}, t)\,dV = 1 + \frac{2t}{\hbar}\int_{\mathbb{R}^3} \rho_0(\mathbf{x})\,\mathrm{Im}U(\mathbf{x})\,dV + \cdots .$$

As time increases, how does the quantity on the left of this equation behave if $\mathrm{Im}U(\mathbf{x}) < 0$?

Paper 2, Section II

17B Quantum Mechanics

(i) Consider a particle of mass m confined to a one-dimensional potential well of depth $U > 0$ and potential

$$V(x) = \begin{cases} -U, & |x| < l \\ 0, & |x| > l. \end{cases}$$

If the particle has energy E where $-U \leqslant E < 0$, show that for even states

$$\alpha \tan \alpha l = \beta$$

where $\alpha = [\frac{2m}{\hbar^2}(U + E)]^{1/2}$ and $\beta = [-\frac{2m}{\hbar^2}E]^{1/2}$.

(ii) A particle of mass m that is incident from the left scatters off a one-dimensional potential given by

$$V(x) = k\delta(x)$$

where $\delta(x)$ is the Dirac delta. If the particle has energy $E > 0$ and $k > 0$, obtain the reflection and transmission coefficients R and T, respectively. Confirm that $R + T = 1$.

For the case $k < 0$ and $E < 0$ show that the energy of the only even parity bound state of the system is given by

$$E = -\frac{mk^2}{2\hbar^2}.$$

Use part (i) to verify this result by taking the limit $U \to \infty,\ l \to 0$ with Ul fixed.

Paper 1, Section I

7H Statistics

Let $x_1, \ldots, x_n$ be independent and identically distributed observations from a distribution with probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda(x-\mu)}, & x \geqslant \mu, \\ 0, & x < \mu, \end{cases}$$

where λ and μ are unknown positive parameters. Let $\beta = \mu + 1/\lambda$. Find the maximum likelihood estimators $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\beta}$.

Determine for each of $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\beta}$ whether or not it has a positive bias.

Paper 2, Section I

8H Statistics

State and prove the Rao–Blackwell theorem.

Individuals in a population are independently of three types $\{0, 1, 2\}$, with unknown probabilities p_0, p_1, p_2 where $p_0 + p_1 + p_2 = 1$. In a random sample of n people the ith person is found to be of type $x_i \in \{0, 1, 2\}$.

Show that an unbiased estimator of $\theta = p_0 p_1 p_2$ is

$$\hat{\theta} = \begin{cases} 1, & \text{if } (x_1, x_2, x_3) = (0, 1, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that n_i of the individuals are of type i. Find an unbiased estimator of θ, say θ^*, such that $\text{var}(\theta^*) < \theta(1 - \theta)$.

Paper 4, Section II

19H Statistics

Explain the notion of a sufficient statistic.

Suppose X is a random variable with distribution F taking values in $\{1, \ldots, 6\}$, with $P(X = i) = p_i$. Let $x_1, \ldots, x_n$ be a sample from F. Suppose n_i is the number of these x_j that are equal to i. Use a factorization criterion to explain why $(n_1, \ldots, n_6)$ is sufficient for $\theta = (p_1, \ldots, p_6)$.

Let H_0 be the hypothesis that $p_i = 1/6$ for all i. Derive the statistic of the generalized likelihood ratio test of H_0 against the alternative that this is not a good fit.

Assuming that $n_i \approx n/6$ when H_0 is true and n is large, show that this test can be approximated by a chi-squared test using a test statistic

$$T = -n + \frac{6}{n} \sum_{i=1}^{6} n_i^2 .$$

Suppose $n = 100$ and $T = 8.12$. Would you reject H_0? Explain your answer.

Paper 1, Section II

19H Statistics

Consider the general linear model $Y = X\theta + \epsilon$ where X is a known $n \times p$ matrix, θ is an unknown $p \times 1$ vector of parameters, and ϵ is an $n \times 1$ vector of independent $N(0, \sigma^2)$ random variables with unknown variance σ^2. Assume the $p \times p$ matrix $X^T X$ is invertible. Let

$$\hat{\theta} = (X^T X)^{-1} X^T Y$$
$$\hat{\epsilon} = Y - X\hat{\theta}.$$

What are the distributions of $\hat{\theta}$ and $\hat{\epsilon}$? Show that $\hat{\theta}$ and $\hat{\epsilon}$ are uncorrelated.

Four apple trees stand in a 2×2 rectangular grid. The annual yield of the tree at coordinate (i, j) conforms to the model

$$y_{ij} = \alpha_i + \beta x_{ij} + \epsilon_{ij}, \quad i, j \in \{1, 2\},$$

where x_{ij} is the amount of fertilizer applied to tree (i, j), α_1, α_2 may differ because of varying soil across rows, and the ϵ_{ij} are $N(0, \sigma^2)$ random variables that are independent of one another and from year to year. The following two possible experiments are to be compared:

$$\mathrm{I}: \ (x_{ij}) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathrm{II}: \ (x_{ij}) = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}.$$

Represent these as general linear models, with $\theta = (\alpha_1, \alpha_2, \beta)$. Compare the variances of estimates of β under I and II.

With II the following yields are observed:

$$(y_{ij}) = \begin{pmatrix} 100 & 300 \\ 600 & 400 \end{pmatrix}.$$

Forecast the total yield that will be obtained next year if no fertilizer is used. What is the 95% predictive interval for this yield?

Paper 3, Section II

20H Statistics

Suppose x_1 is a single observation from a distribution with density f over $[0,1]$. It is desired to test $H_0 : f(x) = 1$ against $H_1 : f(x) = 2x$.

Let $\delta : [0,1] \to \{0,1\}$ define a test by $\delta(x_1) = i \iff$ 'accept H_i'. Let $\alpha_i(\delta) = P(\delta(x_1) = 1 - i \mid H_i)$. State the Neyman-Pearson lemma using this notation.

Let δ be the best test of size 0.10. Find δ and $\alpha_1(\delta)$.

Consider now $\delta : [0,1] \to \{0,1,\star\}$ where $\delta(x_1) = \star$ means 'declare the test to be inconclusive'. Let $\gamma_i(\delta) = P(\delta(x) = \star \mid H_i)$. Given prior probabilities π_0 for H_0 and $\pi_1 = 1 - \pi_0$ for H_1, and some w_0, w_1, let

$$\operatorname{cost}(\delta) = \pi_0\big(w_0\alpha_0(\delta) + \gamma_0(\delta)\big) + \pi_1\big(w_1\alpha_1(\delta) + \gamma_1(\delta)\big).$$

Let $\delta^*(x_1) = i \iff x_1 \in A_i$, where $A_0 = [0, 0.5)$, $A_\star = [0.5, 0.6)$, $A_1 = [0.6, 1]$. Prove that for each value of $\pi_0 \in (0,1)$ there exist w_0, w_1 (depending on π_0) such that $\operatorname{cost}(\delta^*) = \min_\delta \operatorname{cost}(\delta)$. [*Hint*: $w_0 = 1 + 2(0.6)(\pi_1/\pi_0)$.]

Hence prove that if δ is any test for which

$$\alpha_i(\delta) \leqslant \alpha_i(\delta^*), \quad i = 0, 1$$

then $\gamma_0(\delta) \geqslant \gamma_0(\delta^*)$ and $\gamma_1(\delta) \geqslant \gamma_1(\delta^*)$.

Paper 1, Section I

4A Variational Principles

(a) Define what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be convex. Assuming g'' exists, state an equivalent condition. Let $f(x) = x \log x$, defined on $x > 0$. Show that $f(x)$ is convex.

(b) Find the Legendre transform $f^*(p)$ of $f(x) = x \log x$. State the domain of $f^*(p)$. Without further calculation, explain why $(f^*)^* = f$ in this case.

Paper 3, Section I

6A Variational Principles

A cylindrical drinking cup has thin curved sides with density ρ per unit area, and a disk-shaped base with density $k\rho$ per unit area. The cup has capacity to hold a fixed volume V of liquid. Use the method of Lagrange multipliers to find the minimum mass of the cup.

Paper 2, Section II

15A Variational Principles

Starting from the Euler–Lagrange equation, show that a condition for

$$\int f(y, y')dx$$

to be stationary is

$$f - y'\frac{\partial f}{\partial y'} = \text{constant.}$$

In the half-plane $y > 0$, light has speed $c(y) = y + c_0$ where $c_0 > 0$. Find the equation for a light ray between $(-a, 0)$ and $(a, 0)$. Sketch the solution.

Paper 4, Section II

16A Variational Principles

Derive the Euler–Lagrange equation for the integral

$$\int_a^b f(x, y, y', y'')\, dx$$

where prime denotes differentiation with respect to x, and both y and y' are specified at $x = a, b$.

Find $y(x)$ that extremises the integral

$$\int_0^\pi \left(y + \frac{1}{2}y^2 - \frac{1}{2}y''^2 \right) dx$$

subject to $y(0) = -1$, $y'(0) = 0$, $y(\pi) = \cosh\pi$ and $y'(\pi) = \sinh\pi$.

Show that your solution is a global maximum. You may use the result that

$$\int_0^\pi \phi^2(x)dx \leqslant \int_0^\pi \phi'^2(x)dx$$

for any (suitably differentiable) function ϕ which satisfies $\phi(0) = 0$ and $\phi(\pi) = 0$.