UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday, 6 June, 2012 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Define the determinant $\det A$ of an $n \times n$ real matrix A. Suppose that X is a matrix with block form

$$X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A, B and C are matrices of dimensions $n \times n$, $n \times m$ and $m \times m$ respectively. Show that $\det X = (\det A)(\det C)$.

2G Groups, Rings and Modules

What does it mean to say that the finite group G *acts* on the set Ω?

By considering an action of the symmetry group of a regular tetrahedron on a set of pairs of edges, show there is a surjective homomorphism $S_4 \to S_3$.

[You may assume that the symmetric group S_n is generated by transpositions.]

3E Analysis II

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. What does it mean to say that f is differentiable at a point $(x, y) \in \mathbb{R}^2$? Prove directly from this definition, that if f is differentiable at (x, y), then f is continuous at (x, y).

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function:

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } x \text{ and } y \text{ are rational} \\ 0 & \text{otherwise.} \end{cases}$$

For which points $(x, y) \in \mathbb{R}^2$ is f differentiable? Justify your answer.

4F Metric and Topological Spaces

For each case below, determine whether the given metrics d_1 and d_2 induce the same topology on X. Justify your answers.

(i) $X = \mathbb{R}^2$, $d_1((x_1, y_1), (x_2, y_2)) = \sup\{|x_1 - x_2|, |y_1 - y_2|\}$

$$d_2((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

(ii) $X = C[0,1]$, $d_1(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$

$$d_2(f, g) = \int_0^1 |f(t) - g(t)|\, dt.$$

5C Methods

Using the method of characteristics, obtain a solution to the equation

$$u_x + 2xu_y = y$$

subject to the Cauchy data $u(0, y) = 1 + y^2$ for $-\frac{1}{2} < y < \frac{1}{2}$.

Sketch the characteristics and specify the greatest region of the plane in which a unique solution exists.

6B Electromagnetism

Write down the expressions for a general, time-dependent electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in terms of a vector potential $\mathbf{A}$ and scalar potential ϕ. What is meant by a gauge transformation of $\mathbf{A}$ and ϕ? Show that $\mathbf{E}$ and $\mathbf{B}$ are unchanged under a gauge transformation.

A plane electromagnetic wave has vector and scalar potentials

$$\begin{aligned} \mathbf{A}(\mathbf{x}, t) &= \mathbf{A}_0\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \\ \phi(\mathbf{x}, t) &= \phi_0\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \end{aligned}$$

where $\mathbf{A}_0$ and ϕ_0 are constants. Show that $(\mathbf{A}_0, \phi_0)$ can be modified to $(\mathbf{A}_0 + \mu\mathbf{k}, \phi_0 + \mu\omega)$ by a gauge transformation. What choice of μ leads to the modified $\mathbf{A}(\mathbf{x}, t)$ satisfying the Coulomb gauge condition $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$?

7A Fluid Dynamics

Starting from Euler's equation for the motion of an inviscid fluid, derive the vorticity equation in the form

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} .$$

Deduce that an initially irrotational flow remains irrotational.

Consider a plane flow that at time $t = 0$ is described by the streamfunction

$$\psi = x^2 + y^2 .$$

Calculate the vorticity everywhere at times $t > 0$.

8H Statistics

Let the sample $x = (x_1, \ldots, x_n)$ have likelihood function $f(x; \theta)$. What does it mean to say $T(x)$ is a sufficient statistic for θ?

Show that if a certain factorization criterion is satisfied then T is sufficient for θ.

Suppose that T is sufficient for θ and there exist two samples, x and y, for which $T(x) \neq T(y)$ and $f(x; \theta)/f(y; \theta)$ does not depend on θ. Let

$$T_1(z) = \begin{cases} T(z) & z \neq y \\ T(x) & z = y. \end{cases}$$

Show that T_1 is also sufficient for θ.

Explain why T is not minimally sufficient for θ.

9H Optimization

Consider the two-player zero-sum game with payoff matrix

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 3 & 4 & 5 \\ 6 & 0 & 6 \end{pmatrix} .$$

Express the problem of finding the column player's optimal strategy as a linear programming problem in which $x_1 + x_2 + x_3$ is to be maximized subject to some constraints.

Solve this problem using the simplex algorithm and find the optimal strategy for the column player.

Find also, from the final tableau you obtain, both the value of the game and the row player's optimal strategy.

SECTION II

10F Linear Algebra

(i) Define the transpose of a matrix. If V and W are finite-dimensional real vector spaces, define the dual of a linear map $T : V \to W$. How are these two notions related?

Now suppose V and W are finite-dimensional inner product spaces. Use the inner product on V to define a linear map $V \to V^*$ and show that it is an isomorphism. Define the adjoint of a linear map $T : V \to W$. How are the adjoint of T and its dual related? If A is a matrix representing T, under what conditions is the adjoint of T represented by the transpose of A?

(ii) Let $V = C[0,1]$ be the vector space of continuous real-valued functions on $[0,1]$, equipped with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt\,.$$

Let $T : V \to V$ be the linear map

$$Tf(t) = \int_0^t f(s)\,ds\,.$$

What is the adjoint of T?

11G Groups, Rings and Modules

State Gauss's Lemma. State Eisenstein's irreducibility criterion.

(i) By considering a suitable substitution, show that the polynomial $1 + X^3 + X^6$ is irreducible over $\mathbb{Q}$.

(ii) By working in $\mathbb{Z}_2[X]$, show that the polynomial $1 - X^2 + X^5$ is irreducible over $\mathbb{Q}$.

12E Analysis II

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Fix $a \in \mathbb{R}^n$ and prove that the following two statements are equivalent:

(i) Given $\varepsilon > 0$ there is $\delta > 0$ such that $\|f(x) - f(a)\| < \varepsilon$ whenever $\|x - a\| < \delta$ (we use the standard norm in Euclidean space).

(ii) $f(x_n) \to f(a)$ for any sequence $x_n \to a$.

We say that f is continuous if (i) (or equivalently (ii)) holds for every $a \in \mathbb{R}^n$.

Let E and F be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For $f : \mathbb{R}^n \to \mathbb{R}^m$ as above, determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate.

(a) If $f^{-1}(F)$ is closed whenever F is closed, then f is continuous.

(b) If f is continuous, then $f^{-1}(F)$ is closed whenever F is closed.

(c) If f is continuous, then $f(E)$ is open whenever E is open.

(d) If f is continuous, then $f(E)$ is bounded whenever E is bounded.

(e) If f is continuous and $f^{-1}(F)$ is bounded whenever F is bounded, then $f(E)$ is closed whenever E is closed.

13A Complex Analysis or Complex Methods

By a suitable choice of contour show that, for $-1 < \alpha < 1$,

$$\int_0^\infty \frac{x^\alpha}{1+x^2}\, \mathrm{d}x \quad = \quad \frac{\pi}{2\cos(\alpha\pi/2)}\,.$$

14G Geometry

Let S be a closed surface, equipped with a triangulation. Define the Euler characteristic $\chi(S)$ of S. How does $\chi(S)$ depend on the triangulation?

Let V, E and F denote the number of vertices, edges and faces of the triangulation. Show that $2E = 3F$.

Suppose now the triangulation is *tidy*, meaning that it has the property that no two vertices are joined by more than one edge. Deduce that V satisfies

$$V \geqslant \frac{7 + \sqrt{49 - 24\chi(S)}}{2}\,.$$

Hence compute the minimal number of vertices of a tidy triangulation of the real projective plane. [*Hint: it may be helpful to consider the icosahedron as a triangulation of the sphere* S^2.]

15B Variational Principles

(i) A two-dimensional oscillator has action

$$S = \int_{t_0}^{t_1} \left\{ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\omega^2 y^2 \right\} dt \,.$$

Find the equations of motion as the Euler-Lagrange equations associated to S, and use them to show that

$$J = \dot{x}y - \dot{y}x$$

is conserved. Write down the general solution of the equations of motion in terms of $\sin \omega t$ and $\cos \omega t$, and evaluate J in terms of the coefficients which arise in the general solution.

(ii) Another kind of oscillator has action

$$\widetilde{S} = \int_{t_0}^{t_1} \left\{ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{4}\alpha x^4 - \frac{1}{2}\beta x^2 y^2 - \frac{1}{4}\alpha y^4 \right\} dt \,,$$

where α and β are real constants. Find the equations of motion and use these to show that in general $J = \dot{x}y - \dot{y}x$ is not conserved. Find the special value of the ratio β/α for which J is conserved. Explain what is special about the action $\widetilde{S}$ in this case, and state the interpretation of J.

16C Methods

Consider the linear differential operator $\mathcal{L}$ defined by

$$\mathcal{L}y := -\frac{d^2y}{dx^2} + y$$

on the interval $0 \leqslant x < \infty$. Given the boundary conditions $y(0) = 0$ and $\lim_{x\to\infty} y(x) = 0$, find the Green's function $G(x, \xi)$ for $\mathcal{L}$ with these boundary conditions. Hence, or otherwise, obtain the solution of

$$\mathcal{L}y = \begin{cases} 1, & 0 \leqslant x \leqslant \mu \\ 0, & \mu < x < \infty \end{cases}$$

subject to the above boundary conditions, where μ is a positive constant. Show that your piecewise solution is continuous at $x = \mu$ and has the value

$$y(\mu) = \frac{1}{2}(1 + e^{-2\mu} - 2e^{-\mu}) \,.$$

17C Quantum Mechanics

Consider a quantum mechanical particle in a one-dimensional potential $V(x)$, for which $V(x) = V(-x)$. Prove that when the energy eigenvalue E is non-degenerate, the energy eigenfunction $\chi(x)$ has definite parity.

Now assume the particle is in the double potential well

$$V(x) = \begin{cases} U\,, & 0 \leqslant |x| \leqslant l_1 \\ 0\,, & l_1 < |x| \leqslant l_2 \\ \infty\,, & l_2 < |x|\,, \end{cases}$$

where $0 < l_1 < l_2$ and $0 < E < U$ (U being large and positive). Obtain general expressions for the even parity energy eigenfunctions $\chi^+(x)$ in terms of trigonometric and hyperbolic functions. Show that

$$-\tan[k(l_2 - l_1)] = \frac{k}{\kappa}\coth(\kappa l_1)\,,$$

where $k^2 = \dfrac{2mE}{\hbar^2}$ and $\kappa^2 = \dfrac{2m(U-E)}{\hbar^2}$.

18B Electromagnetism

A straight wire has n mobile, charged particles per unit length, each of charge q. Assuming the charges all move with velocity v along the wire, show that the current is $I = nqv$.

Using the Lorentz force law, show that if such a current-carrying wire is placed in a uniform magnetic field of strength B perpendicular to the wire, then the force on the wire, per unit length, is BI.

Consider two infinite parallel wires, with separation L, carrying (in the same sense of direction) positive currents I_1 and I_2, respectively. Find the force per unit length on each wire, determining both its magnitude and direction.

19D Numerical Analysis

Let $\{P_n\}_{n=0}^{\infty}$ be the sequence of monic polynomials of degree n orthogonal on the interval $[-1,1]$ with respect to the weight function w.

Prove that each P_n has n distinct zeros in the interval $(-1,1)$.

Let $P_0(x)=1$, $P_1(x)=x-a_1$, and let P_n satisfy the following three-term recurrence relation:

$$P_n(x)=(x-a_n)P_{n-1}(x)-b_n^2P_{n-2}(x)\,, \qquad n\geqslant 2\,.$$

Set

$$A_n=\begin{bmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-1} & a_{n-1} & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{bmatrix}.$$

Prove that $P_n(x)=\det(xI-A_n)$, $n\geqslant 1$, and deduce that all the eigenvalues of A_n are distinct and reside in $(-1,1)$.

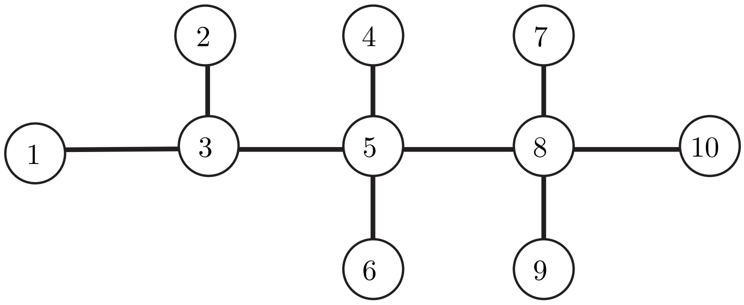
20H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be the symmetric random walk on vertices of a connected graph. At each step this walk jumps from the current vertex to a neighbouring vertex, choosing uniformly amongst them. Let $T_i = \inf\{n \geqslant 1 : X_n = i\}$. For each $i \neq j$ let $q_{ij} = P(T_j < T_i \mid X_0 = i)$ and $m_{ij} = E(T_j \mid X_0 = i)$. Stating any theorems that you use:

(i) Prove that the invariant distribution π satisfies detailed balance.

(ii) Use reversibility to explain why $\pi_i q_{ij} = \pi_j q_{ji}$ for all i, j.

Consider a symmetric random walk on the graph shown below.



(iii) Find m_{33}.

(iv) The removal of any edge (i, j) leaves two disjoint components, one which includes i and one which includes j. Prove that $m_{ij} = 1 + 2e_{ij}(i)$, where $e_{ij}(i)$ is the number of edges in the component that contains i.

(v) Show that $m_{ij} + m_{ji} \in \{18, 36, 54, 72\}$ for all $i \neq j$.

END OF PAPER