UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2012

List of Courses

Paper 1, Section I

3E Analysis I

What does it mean to say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous* at $x_0 \in \mathbb{R}$?

Give an example of a continuous function $f\colon (0,1] \to \mathbb{R}$ which is bounded but attains neither its upper bound nor its lower bound.

The function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and non-negative, and satisfies $f(x) \to 0$ as $x \to \infty$ and $f(x) \to 0$ as $x \to -\infty$. Show that f is bounded above and attains its upper bound.

[Standard results about continuous functions on closed bounded intervals may be used without proof if clearly stated.]

Paper 1, Section I

4F Analysis I

Let $f, g : [0,1] \to \mathbb{R}$ be continuous functions with $g(x) \geqslant 0$ for $x \in [0,1]$. Show that

$$\int_0^1 f(x)g(x)\,dx \leqslant M \int_0^1 g(x)\,dx\,,$$

where $M = \sup\{|f(x)| : x \in [0,1]\}$.

Prove there exists $\alpha \in [0,1]$ such that

$$\int_0^1 f(x)g(x)\,dx = f(\alpha) \int_0^1 g(x)\,dx\,.$$

[Standard results about continuous functions and their integrals may be used without proof, if clearly stated.]

Paper 1, Section II

9E Analysis I

(a) What does it mean to say that the sequence (x_n) of real numbers *converges* to $\ell \in \mathbb{R}$?

Suppose that $(y_n^{(1)})$, $(y_n^{(2)})$,..., $(y_n^{(k)})$ are sequences of real numbers converging to the same limit ℓ. Let (x_n) be a sequence such that for every n,

$$x_n \in \{y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(k)}\}\,.$$

Show that (x_n) also converges to ℓ.

Find a collection of sequences $(y_n^{(j)})$, $j = 1, 2, \ldots$ such that for every j, $(y_n^{(j)}) \to \ell$ but the sequence (x_n) defined by $x_n = y_n^{(n)}$ diverges.

(b) Let a, b be real numbers with $0 < a < b$. Sequences (a_n), (b_n) are defined by $a_1 = a$, $b_1 = b$ and

$$\text{for all } n \geqslant 1, \quad a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}\,.$$

Show that (a_n) and (b_n) converge to the same limit.

Paper 1, Section II

10D Analysis I

Let (a_n) be a sequence of reals.

(i) Show that if the sequence $(a_{n+1} - a_n)$ is convergent then so is the sequence $\left(\frac{a_n}{n}\right)$.

(ii) Give an example to show the sequence $\left(\frac{a_n}{n}\right)$ being convergent does not imply that the sequence $(a_{n+1} - a_n)$ is convergent.

(iii) If $a_{n+k} - a_n \to 0$ as $n \to \infty$ for each positive integer k, does it follow that (a_n) is convergent? Justify your answer.

(iv) If $a_{n+f(n)} - a_n \to 0$ as $n \to \infty$ for every function f from the positive integers to the positive integers, does it follow that (a_n) is convergent? Justify your answer.

Paper 1, Section II
11D Analysis I

Let f be a continuous function from $(0,1)$ to $(0,1)$ such that $f(x) < x$ for every $0 < x < 1$. We write f^n for the n-fold composition of f with itself (so for example $f^2(x) = f(f(x))$).

(i) Prove that for every $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$.

(ii) Must it be the case that for every $\epsilon > 0$ there exists n with the property that $f^n(x) < \epsilon$ for all $0 < x < 1$? Justify your answer.

Now suppose that we remove the condition that f be continuous.

(iii) Give an example to show that it need not be the case that for every $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$.

(iv) Must it be the case that for *some* $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$? Justify your answer.

Paper 1, Section II
12F Analysis I

(a) (i) State the ratio test for the convergence of a real series with positive terms.

(ii) Define the radius of convergence of a real power series $\sum_{n=0}^{\infty} a_n x^n$.

(iii) Prove that the real power series $f(x) = \sum_n a_n x^n$ and $g(x) = \sum_n (n+1)a_{n+1}x^n$ have equal radii of convergence.

(iv) State the relationship between $f(x)$ and $g(x)$ within their interval of convergence.

(b) (i) Prove that the real series

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

have radius of convergence ∞.

(ii) Show that they are differentiable on the real line $\mathbb{R}$, with $f' = -g$ and $g' = f$, and deduce that $f(x)^2 + g(x)^2 = 1$.

[You may use, without proof, general theorems about differentiating within the interval of convergence, provided that you give a clear statement of any such theorem.]

Paper 2, Section I

1A Differential Equations

Find two linearly independent solutions of

$$y'' + 4y' + 4y = 0\,.$$

Find the solution in $x \geqslant 0$ of

$$y'' + 4y' + 4y = e^{-2x}\,,$$

subject to $y = y' = 0$ at $x = 0$.

Paper 2, Section I

2A Differential Equations

Find the constant solutions (those with $u_{n+1} = u_n$) of the discrete equation

$$u_{n+1} = \tfrac{1}{2}u_n\left(1 + u_n\right)\,,$$

and determine their stability.

Paper 2, Section II

5A Differential Equations

Find the first three non-zero terms in the series solutions $y_1(x)$ and $y_2(x)$ for the differential equation

$$x^2 y'' - 2xy' + (2 - x^2)y = 0\,,$$

that satisfy

$$\begin{aligned} y_1'(0) = a \quad &\text{and} \quad y_1''(0) = 0\,,\\ y_2'(0) = 0 \quad &\text{and} \quad y_2''(0) = 2b\,. \end{aligned}$$

Identify these solutions in closed form.

Paper 2, Section II
6A Differential Equations
Consider the function

$$V(x,y) = x^4 - x^2 + 2xy + y^2.$$

Find the critical (stationary) points of $V(x,y)$. Determine the type of each critical point. Sketch the contours of $V(x,y) = \text{constant}$.

Now consider the coupled differential equations

$$\frac{dx}{dt} = -\frac{\partial V}{\partial x}, \qquad \frac{dy}{dt} = -\frac{\partial V}{\partial y}.$$

Show that $V(x(t), y(t))$ is a non-increasing function of t. If $x = 1$ and $y = -\frac{1}{2}$ at $t = 0$, where does the solution tend to as $t \to \infty$?

Paper 2, Section II
7A Differential Equations
Find the solution to the system of equations

$$\begin{aligned} \frac{dx}{dt} + \frac{-4x + 2y}{t} &= -9, \\ \frac{dy}{dt} + \frac{x - 5y}{t} &= 3 \end{aligned}$$

in $t \geqslant 1$ subject to

$$x = 0 \quad \text{and} \quad y = 0 \quad \text{at} \quad t = 1.$$

[*Hint: powers of* t.]

Paper 2, Section II

8A Differential Equations

Consider the second-order differential equation for $y(t)$ in $t \geqslant 0$

$$\ddot{y} + 2k\dot{y} + (k^2 + \omega^2)y = f(t)\,. \qquad (*)$$

(i) For $f(t) = 0$, find the general solution $y_1(t)$ of $(*)$.

(ii) For $f(t) = \delta(t-a)$ with $a > 0$, find the solution $y_2(t,a)$ of $(*)$ that satisfies $y = 0$ and $\dot{y} = 0$ at $t = 0$.

(iii) For $f(t) = H(t-b)$ with $b > 0$, find the solution $y_3(t,b)$ of $(*)$ that satisfies $y = 0$ and $\dot{y} = 0$ at $t = 0$.

(iv) Show that

$$y_2(t,b) = -\frac{\partial y_3}{\partial b}\,.$$

Paper 4, Section I

3B Dynamics and Relativity

Two particles of masses m_1 and m_2 have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Particle 2 exerts a force $\mathbf{F}_{12}(\mathbf{r})$ on particle 1 (where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$) and there are no external forces.

Prove that the centre of mass of the two-particle system will move at constant speed along a straight line.

Explain how the two-body problem of determining the motion of the system may be reduced to that of a single particle moving under the force $\mathbf{F}_{12}$.

Suppose now that $m_1 = m_2 = m$ and that

$$\mathbf{F}_{12} = -\frac{Gm^2}{r^3}\mathbf{r}$$

is gravitational attraction. Let C be a circle fixed in space. Is it possible (by suitable choice of initial conditions) for the two particles to be traversing C at the same constant angular speed? Give a brief reason for your answer.

Paper 4, Section I

4B Dynamics and Relativity

Let S and S' be inertial frames in 2-dimensional spacetime with coordinate systems (t, x) and (t', x') respectively. Suppose that S' moves with positive velocity v relative to S and the spacetime origins of S and S' coincide. Write down the Lorentz transformation relating the coordinates of any event relative to the two frames.

Show that events which occur simultaneously in S are not generally seen to be simultaneous when viewed in S'.

In S two light sources A and B are at rest and placed a distance d apart. They simultaneously each emit a photon in the positive x direction. Show that in S' the photons are separated by a constant distance $d\sqrt{\dfrac{c+v}{c-v}}$.

Paper 4, Section II

9B Dynamics and Relativity

Let (r, θ) be polar coordinates in the plane. A particle of mass m moves in the plane under an attractive force of $mf(r)$ towards the origin O. You may assume that the acceleration $\mathbf{a}$ is given by

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta}$$

where $\hat{\mathbf{r}}$ and $\hat{\theta}$ are the unit vectors in the directions of increasing r and θ respectively, and the dot denotes d/dt.

(a) Show that $l = r^2\dot{\theta}$ is a constant of the motion. Introducing $u = 1/r$ show that $\dot{r} = -l\dfrac{du}{d\theta}$ and derive the geometric orbit equation

$$l^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = f\left(\frac{1}{u}\right).$$

(b) Suppose now that

$$f(r) = \frac{3r + 9}{r^3}$$

and that initially the particle is at distance $r_0 = 1$ from O, moving with speed $v_0 = 4$ in a direction making angle $\pi/3$ with the radial vector pointing towards O.

Show that $l = 2\sqrt{3}$ and find u as a function of θ. Hence or otherwise show that the particle returns to its original position after one revolution about O and then flies off to infinity.

Paper 4, Section II

10B Dynamics and Relativity

For any frame S and vector $\mathbf{A}$, let $\left[\frac{d\mathbf{A}}{dt}\right]_S$ denote the derivative of $\mathbf{A}$ relative to S. A frame of reference S' rotates with constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame S and the two frames have a common origin O. [You may assume that for any vector $\mathbf{A}$, $\left[\frac{d\mathbf{A}}{dt}\right]_S = \left[\frac{d\mathbf{A}}{dt}\right]_{S'} + \boldsymbol{\omega} \times \mathbf{A}$.]

(a) If $\mathbf{r}(t)$ is the position vector of a point P from O, show that

$$\left[\frac{d^2\mathbf{r}}{dt^2}\right]_S = \left[\frac{d^2\mathbf{r}}{dt^2}\right]_{S'} + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{v}' = \left[\frac{d\mathbf{r}}{dt}\right]_{S'}$ is the velocity in S'.

Suppose now that $\mathbf{r}(t)$ is the position vector of a particle of mass m moving under a conservative force $\mathbf{F} = -\nabla\phi$ and a force $\mathbf{G}$ that is always orthogonal to the velocity $\mathbf{v}'$ in S'. Show that the quantity

$$E = \frac{1}{2}m\mathbf{v}'.\mathbf{v}' + \phi - \frac{m}{2}(\boldsymbol{\omega} \times \mathbf{r}).(\boldsymbol{\omega} \times \mathbf{r})$$

is a constant of the motion. [You may assume that $\nabla\left[(\boldsymbol{\omega} \times \mathbf{r}).(\boldsymbol{\omega} \times \mathbf{r})\right] = -2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$.]

(b) A bead slides on a frictionless circular hoop of radius a which is forced to rotate with constant angular speed ω about a vertical diameter. Let $\theta(t)$ denote the angle between the line from the centre of the hoop to the bead and the downward vertical. Using the results of (a), or otherwise, show that

$$\ddot{\theta} + \left(\frac{g}{a} - \omega^2 \cos\theta\right)\sin\theta = 0.$$

Deduce that if $\omega^2 > g/a$ there are two equilibrium positions $\theta = \theta_0$ off the axis of rotation, and show that these are stable equilibria.

Paper 4, Section II

11B Dynamics and Relativity

(a) State the parallel axis theorem for moments of inertia.

(b) A uniform circular disc D of radius a and total mass m can turn frictionlessly about a fixed horizontal axis that passes through a point A on its circumference and is perpendicular to its plane. Initially the disc hangs at rest (in constant gravity g) with its centre O being vertically below A. Suppose the disc is disturbed and executes free oscillations. Show that the period of small oscillations is $2\pi\sqrt{\dfrac{3a}{2g}}$.

(c) Suppose now that the disc is released from rest when the radius OA is vertical with O directly above A. Find the angular velocity and angular acceleration of O about A when the disc has turned through angle θ. Let $\mathbf{R}$ denote the reaction force at A on the disc. Find the acceleration of the centre of mass of the disc. Hence, or otherwise, show that the component of $\mathbf{R}$ parallel to OA is $mg(7\cos\theta - 4)/3$.

Paper 4, Section II

12B Dynamics and Relativity

(a) Define the 4-momentum $\mathbf{P}$ of a particle of rest mass m and 3-velocity $\mathbf{v}$, and the 4-momentum of a photon of frequency ν (having zero rest mass) moving in the direction of the unit vector $\mathbf{e}$.

Show that if $\mathbf{P}_1$ and $\mathbf{P}_2$ are timelike future-pointing 4-vectors then $\mathbf{P}_1.\mathbf{P}_2 \geqslant 0$ (where the dot denotes the Lorentz-invariant scalar product). Hence or otherwise show that the law of conservation of 4-momentum forbids a photon to spontaneously decay into an electron-positron pair. [Electrons and positrons have equal rest masses $m > 0$.]

(b) In the laboratory frame an electron travelling with velocity $\mathbf{u}$ collides with a positron at rest. They annihilate, producing two photons of frequencies ν_1 and ν_2 that move off at angles θ_1 and θ_2 to $\mathbf{u}$, in the directions of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively. By considering 4-momenta in the laboratory frame, or otherwise, show that

$$\frac{1+\cos(\theta_1+\theta_2)}{\cos\theta_1+\cos\theta_2} = \sqrt{\frac{\gamma-1}{\gamma+1}}$$

where $\gamma = \left(1 - \dfrac{u^2}{c^2}\right)^{-1/2}$.

Paper 3, Section I

1E Groups

State Lagrange's Theorem. Deduce that if G is a finite group of order n, then the order of every element of G is a divisor of n.

Let G be a group such that, for every $g \in G$, $g^2 = e$. Show that G is abelian. Give an example of a non-abelian group in which every element g satisfies $g^4 = e$.

Paper 3, Section I

2E Groups

What is a *cycle* in the symmetric group S_n? Show that a cycle of length p and a cycle of length q in S_n are conjugate if and only if $p = q$.

Suppose that p is odd. Show that any two p-cycles in A_{p+2} are conjugate. Are any two 3-cycles in A_4 conjugate? Justify your answer.

Paper 3, Section II

5E Groups

(i) State and prove the Orbit-Stabilizer Theorem.

Show that if G is a finite group of order n, then G is isomorphic to a subgroup of the symmetric group S_n.

(ii) Let G be a group acting on a set X with a single orbit, and let H be the stabilizer of some element of X. Show that the homomorphism $G \to \mathrm{Sym}(X)$ given by the action is injective if and only if the intersection of all the conjugates of H equals $\{e\}$.

(iii) Let Q_8 denote the quaternion group of order 8. Show that for every $n < 8$, Q_8 is not isomorphic to a subgroup of S_n.

Paper 3, Section II

6E Groups

Let G be $SL_2(\mathbb{R})$, the groups of *real* 2×2 matrices of determinant 1, acting on $\mathbb{C} \cup \{\infty\}$ by Möbius transformations.

For each of the points 0, i, $-i$, compute its stabilizer and its orbit under the action of G. Show that G has exactly 3 orbits in all.

Compute the orbit of i under the subgroup

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a,\ b,\ d \in \mathbb{R},\ ad = 1 \right\} \subset G\,.$$

Deduce that every element g of G may be expressed in the form $g = hk$ where $h \in H$ and for some $\theta \in \mathbb{R}$,

$$k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\,.$$

How many ways are there of writing g in this form?

Paper 3, Section II

7E Groups

Let $\mathbb{F}_p$ be the set of (residue classes of) integers mod p, and let

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\, b,\, c,\, d \in \mathbb{F}_p,\ ad - bc \neq 0 \right\}$$

Show that G is a group under multiplication. [You may assume throughout this question that multiplication of matrices is associative.]

Let X be the set of 2-dimensional column vectors with entries in $\mathbb{F}_p$. Show that the mapping $G \times X \to X$ given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

is a group action.

Let $g \in G$ be an element of order p. Use the orbit-stabilizer theorem to show that there exist $x,\, y \in \mathbb{F}_p$, not both zero, with

$$g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} .$$

Deduce that g is conjugate in G to the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

Paper 3, Section II

8E Groups

Let p be a prime number, and a an integer with $1 \leqslant a \leqslant p-1$. Let G be the Cartesian product

$$G = \{\, (x,u) \mid x \in \{0,1,\ldots,p-2\},\ u \in \{0,1,\ldots,p-1\}\,\}$$

Show that the binary operation

$$(x,u) * (y,v) = (z,w)$$

where

$$z \equiv x + y \pmod{p-1}$$
$$w \equiv a^y u + v \pmod{p}$$

makes G into a group. Show that G is abelian if and only if $a = 1$.

Let H and K be the subsets

$$H = \{\, (x,0) \mid x \in \{0,1,\ldots,p-2\}\,\}, \qquad K = \{\, (0,u) \mid u \in \{0,1,\ldots,p-1\}\,\}$$

of G. Show that K is a normal subgroup of G, and that H is a subgroup which is normal if and only if $a = 1$.

Find a homomorphism from G to another group whose kernel is K.

Paper 4, Section I

1D Numbers and Sets

(i) Find integers x and y such that $18x + 23y = 101$.

(ii) Find an integer x such that $x \equiv 3 \pmod{18}$ and $x \equiv 2 \pmod{23}$.

Paper 4, Section I

2D Numbers and Sets

What is an *equivalence relation* on a set X? If R is an equivalence relation on X, what is an *equivalence class* of R? Prove that the equivalence classes of R form a partition of X.

Let R and S be equivalence relations on a set X. Which of the following are always equivalence relations? Give proofs or counterexamples as appropriate.

(i) The relation V on X given by xVy if both xRy and xSy.

(ii) The relation W on X given by xWy if xRy or xSy.

Paper 4, Section II

5D Numbers and Sets

Let X be a set, and let f and g be functions from X to X. Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) If fg is the identity map then gf is the identity map.

(ii) If $fg = g$ then f is the identity map.

(iii) If $fg = f$ then g is the identity map.

How (if at all) do your answers change if we are given that X is finite?

Determine which sets X have the following property: if f is a function from X to X such that for every $x \in X$ there exists a positive integer n with $f^n(x) = x$, then there exists a positive integer n such that f^n is the identity map. [Here f^n denotes the n-fold composition of f with itself.]

Paper 4, Section II

6D Numbers and Sets

State Fermat's Theorem and Wilson's Theorem.

For which prime numbers p does the equation $x^2 \equiv -1 \pmod{p}$ have a solution? Justify your answer.

For a prime number p, and an integer x that is not a multiple of p, the *order* of x (mod p) is the least positive integer d such that $x^d \equiv 1 \pmod{p}$. Show that if x has order d and also $x^k \equiv 1 \pmod{p}$ then d must divide k.

For a positive integer n, let $F_n = 2^{2^n} + 1$. If p is a prime factor of F_n, determine the order of 2 (mod p). Hence show that the F_n are pairwise coprime.

Show that if p is a prime of the form $4k+3$ then p cannot be a factor of any F_n. Give, with justification, a prime p of the form $4k+1$ such that p is not a factor of any F_n.

Paper 4, Section II

7D Numbers and Sets

Prove that each of the following numbers is irrational:

(i) $\sqrt{2} + \sqrt{3}$

(ii) e

(iii) The real root of the equation $x^3 + 4x - 7 = 0$

(iv) $\log_2 3$.

Paper 4, Section II

8D Numbers and Sets

Show that there is no injection from the power-set of $\mathbb{R}$ to $\mathbb{R}$. Show also that there *is* an injection from $\mathbb{R}^2$ to $\mathbb{R}$.

Let X be the set of all functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x) = x$ for all but finitely many x. Determine whether or not there exists an injection from X to $\mathbb{R}$.

Paper 2, Section I

3F Probability

Given two events A and B with $P(A) > 0$ and $P(B) > 0$, define the conditional probability $P(A \mid B)$.

Show that

$$P(B \mid A) = P(A \mid B)\frac{P(B)}{P(A)}\,.$$

A random number N of fair coins are tossed, and the total number of heads is denoted by H. If $P(N = n) = 2^{-n}$ for $n = 1, 2, \ldots$, find $P(N = n \mid H = 1)$.

Paper 2, Section I

4F Probability

Define the *probability generating function* $G(s)$ of a random variable X taking values in the non-negative integers.

A coin shows heads with probability $p \in (0, 1)$ on each toss. Let N be the number of tosses up to and including the first appearance of heads, and let $k \geqslant 1$. Find the probability generating function of $X = \min\{N, k\}$.

Show that $E(X) = p^{-1}(1 - q^k)$ where $q = 1 - p$.

Paper 2, Section II

9F Probability

(i) Define the *moment generating function* $M_X(t)$ of a random variable X. If X, Y are independent and $a, b \in \mathbb{R}$, show that the moment generating function of $Z = aX + bY$ is $M_X(at)M_Y(bt)$.

(ii) Assume $T > 0$, and $M_X(t) < \infty$ for $|t| < T$. Explain the expansion

$$M_X(t) = 1 + \mu t + \tfrac{1}{2}s^2t^2 + \mathrm{o}(t^2)$$

where $\mu = E(X)$ and $s^2 = E(X^2)$. [You may assume the validity of interchanging expectation and differentiation.]

(iii) Let X, Y be independent, identically distributed random variables with mean 0 and variance 1, and assume their moment generating function M satisfies the condition of part (ii) with $T = \infty$.

Suppose that $X+Y$ and $X-Y$ are independent. Show that $M(2t) = M(t)^3M(-t)$, and deduce that $\psi(t) = M(t)/M(-t)$ satisfies $\psi(t) = \psi(t/2)^2$.

Show that $\psi(h) = 1 + \mathrm{o}(h^2)$ as $h \to 0$, and deduce that $\psi(t) = 1$ for all t.

Show that X and Y are normally distributed.

Paper 2, Section II

10F Probability

(i) Define the distribution function F of a random variable X, and also its density function f assuming F is differentiable. Show that

$$f(x) = -\frac{d}{dx}P(X > x)\,.$$

(ii) Let U, V be independent random variables each with the uniform distribution on $[0, 1]$. Show that

$$P(V^2 > U > x) = \tfrac{1}{3} - x + \tfrac{2}{3}x^{3/2}, \qquad x \in (0, 1)\,.$$

What is the probability that the random quadratic equation $x^2 + 2Vx + U = 0$ has real roots?

Given that the two roots R_1, R_2 of the above quadratic are real, what is the probability that both $|R_1| \leqslant 1$ and $|R_2| \leqslant 1$?

Paper 2, Section II

11F Probability

(i) Let X_n be the size of the n^{th} generation of a branching process with family-size probability generating function $G(s)$, and let $X_0 = 1$. Show that the probability generating function $G_n(s)$ of X_n satisfies $G_{n+1}(s) = G(G_n(s))$ for $n \geqslant 0$.

(ii) Suppose the family-size mass function is $P(X_1 = k) = 2^{-k-1}$, $k = 0, 1, 2, \dots$. Find $G(s)$, and show that

$$G_n(s) = \frac{n-(n-1)s}{n+1-ns} \qquad \text{for } |s| < 1 + \frac{1}{n}\,.$$

Deduce the value of $P(X_n = 0)$.

(iii) Write down the moment generating function of X_n/n. Hence or otherwise show that, for $x \geqslant 0$,

$$P(X_n/n > x \mid X_n > 0) \to e^{-x} \qquad \text{as } n \to \infty\,.$$

[You may use the continuity theorem but, if so, should give a clear statement of it.]

Paper 2, Section II

12F Probability

Let X, Y be independent random variables with distribution functions F_X, F_Y. Show that $U = \min\{X, Y\}$, $V = \max\{X, Y\}$ have distribution functions

$$F_U(u) = 1 - (1 - F_X(u))(1 - F_Y(u))\,, \quad F_V(v) = F_X(v)F_Y(v)\,.$$

Now let X, Y be independent random variables, each having the exponential distribution with parameter 1. Show that U has the exponential distribution with parameter 2, and that $V - U$ is independent of U.

Hence or otherwise show that V has the same distribution as $X + \frac{1}{2}Y$, and deduce the mean and variance of V.

[You may use without proof that X has mean 1 and variance 1.]

Paper 3, Section I

3C Vector Calculus

Define what it means for a differential $P\,dx+Q\,dy$ to be exact, and derive a necessary condition on $P(x,y)$ and $Q(x,y)$ for this to hold. Show that one of the following two differentials is exact and the other is not:

$$y^2\,dx + 2xy\,dy\,,$$
$$y^2\,dx + xy^2\,dy\,.$$

Show that the differential which is not exact can be written in the form $g\,df$ for functions $f(x,y)$ and $g(y)$, to be determined.

Paper 3, Section I

4C Vector Calculus

What does it mean for a second-rank tensor T_{ij} to be *isotropic*? Show that δ_{ij} is isotropic. By considering rotations through $\pi/2$ about the coordinate axes, or otherwise, show that the most general isotropic second-rank tensor in $\mathbb{R}^3$ has the form $T_{ij} = \lambda\delta_{ij}$, for some scalar λ.

Paper 3, Section II

9C Vector Calculus

State Stokes' Theorem for a vector field $\mathbf{B}(\mathbf{x})$ on $\mathbb{R}^3$.

Consider the surface S defined by

$$z = x^2 + y^2, \qquad \frac{1}{9} \leqslant z \leqslant 1.$$

Sketch the surface and calculate the area element $d\mathbf{S}$ in terms of suitable coordinates or parameters. For the vector field

$$\mathbf{B} = (-y^3, x^3, z^3)$$

compute $\nabla \times \mathbf{B}$ and calculate $I = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}$.

Use Stokes' Theorem to express I as an integral over ∂S and verify that this gives the same result.

Paper 3, Section II

10C Vector Calculus

Consider the transformation of variables

$$x = 1 - u, \quad y = \frac{1-v}{1-uv}.$$

Show that the interior of the unit square in the uv plane

$$\{(u,v) : \ 0 < u < 1, \ 0 < v < 1\}$$

is mapped to the interior of the unit square in the xy plane,

$$R = \{(x,y) : \ 0 < x < 1, \ 0 < y < 1\}.$$

[*Hint: Consider the relation between* v *and* y *when* $u = \alpha$, *for* $0 < \alpha < 1$ *constant.*]

Show that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{(1-(1-x)y)^2}{x}.$$

Now let

$$u = \frac{1-t}{1-wt}, \quad v = 1 - w.$$

By calculating

$$\frac{\partial(x,y)}{\partial(t,w)} = \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(t,w)}$$

as a function of x and y, or otherwise, show that

$$\int_R \frac{x(1-y)}{(1-(1-x)y)(1-(1-x^2)y)^2} \, dx \, dy = 1.$$

Paper 3, Section II
11C Vector Calculus

(a) Prove the identity

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}).$$

(b) If $\mathbf{E}$ is an irrotational vector field (i.e. $\nabla\times\mathbf{E} = \mathbf{0}$ everywhere), prove that there exists a scalar potential $\phi(\mathbf{x})$ such that $\mathbf{E} = -\nabla\phi$.

Show that the vector field

$$(xy^2ze^{-x^2z}, -ye^{-x^2z}, \tfrac{1}{2}x^2y^2e^{-x^2z})$$

is irrotational, and determine the corresponding potential ϕ.

Paper 3, Section II
12C Vector Calculus

(i) Let V be a bounded region in $\mathbb{R}^3$ with smooth boundary $S = \partial V$. Show that Poisson's equation in V

$$\nabla^2 u = \rho$$

has at most one solution satisfying $u = f$ on S, where ρ and f are given functions.

Consider the alternative boundary condition $\partial u/\partial n = g$ on S, for some given function g, where n is the outward pointing normal on S. Derive a necessary condition in terms of ρ and g for a solution u of Poisson's equation to exist. Is such a solution unique?

(ii) Find the most general spherically symmetric function $u(r)$ satisfying

$$\nabla^2 u = 1$$

in the region $r = |\mathbf{r}| \leqslant a$ for $a > 0$. Hence in each of the following cases find all possible solutions satisfying the given boundary condition at $r = a$:

(a) $u = 0\,,$

(b) $\frac{\partial u}{\partial n} = 0\,.$

Compare these with your results in part (i).

Paper 1, Section I

1C Vectors and Matrices

(a) Let R be the set of all $z \in \mathbb{C}$ with real part 1. Draw a picture of R and the image of R under the map $z \mapsto e^z$ in the complex plane.

(b) For each of the following equations, find all complex numbers z which satisfy it:

(i) $e^z = e$,

(ii) $(\log z)^2 = -\dfrac{\pi^2}{4}$.

(c) Prove that there is no complex number z satisfying $|z| - z = i$.

Paper 1, Section I

2A Vectors and Matrices

Define what is meant by the terms *rotation*, *reflection*, *dilation* and *shear*. Give examples of real 2×2 matrices representing each of these.

Consider the three 2×2 matrices

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad C = AB\,.$$

Identify the three matrices in terms of your definitions above.

Paper 1, Section II

5C Vectors and Matrices

The equation of a plane Π in $\mathbb{R}^3$ is

$$\mathbf{x} \cdot \mathbf{n} = d\,,$$

where d is a constant scalar and $\mathbf{n}$ is a unit vector normal to Π. What is the distance of the plane from the origin O?

A sphere S with centre $\mathbf{p}$ and radius r satisfies the equation

$$|\mathbf{x} - \mathbf{p}|^2 = r^2\,.$$

Show that the intersection of Π and S contains exactly one point if $|\mathbf{p} \cdot \mathbf{n} - d| = r$.

The tetrahedron $OABC$ is defined by the vectors $\mathbf{a} = \vec{OA}$, $\mathbf{b} = \vec{OB}$, and $\mathbf{c} = \vec{OC}$ with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$. What does the condition $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$ imply about the set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$? A sphere T_r with radius $r > 0$ lies inside the tetrahedron and intersects each of the three faces OAB, OBC, and OCA in exactly one point. Show that the centre P of T_r satisfies

$$\vec{OP} = r\frac{|\mathbf{b} \times \mathbf{c}|\mathbf{a} + |\mathbf{c} \times \mathbf{a}|\mathbf{b} + |\mathbf{a} \times \mathbf{b}|\mathbf{c}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}\,.$$

Given that the vector $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is orthogonal to the plane Ψ of the face ABC, obtain an equation for Ψ. What is the distance of Ψ from the origin?

Paper 1, Section II

6A Vectors and Matrices

Explain why the number of solutions $\mathbf{x}$ of the simultaneous linear equations $A\mathbf{x} = \mathbf{b}$ is 0, 1 or infinity, where A is a real 3×3 matrix and $\mathbf{x}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$. State necessary and sufficient conditions on A and $\mathbf{b}$ for each of these possibilities to hold.

Let A and B be real 3×3 matrices. Give necessary and sufficient conditions on A for there to exist a unique real 3×3 matrix X satisfying $AX = B$.

Find X when

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{pmatrix}$$

Paper 1, Section II
7B Vectors and Matrices

(a) Consider the matrix

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$

Determine whether or not M is diagonalisable.

(b) Prove that if A and B are similar matrices then A and B have the same eigenvalues with the same corresponding algebraic multiplicities.

Is the converse true? Give either a proof (if true) or a counterexample with a brief reason (if false).

(c) State the Cayley-Hamilton theorem for a complex matrix A and prove it in the case when A is a 2×2 diagonalisable matrix.

Suppose that an $n \times n$ matrix B has $B^k = \mathbf{0}$ for some $k > n$ (where $\mathbf{0}$ denotes the zero matrix). Show that $B^n = \mathbf{0}$.

Paper 1, Section II
8B Vectors and Matrices

(a) (i) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

(ii) Show that the quadric $\mathcal{Q}$ in $\mathbb{R}^3$ defined by

$$3x^2 + 2xy + 2y^2 + 2xz + 2z^2 = 1$$

is an ellipsoid. Find the matrix B of a linear transformation of $\mathbb{R}^3$ that will map $\mathcal{Q}$ onto the unit sphere $x^2 + y^2 + z^2 = 1$.

(b) Let P be a real orthogonal matrix. Prove that:

(i) as a mapping of vectors, P preserves inner products;

(ii) if λ is an eigenvalue of P then $|\lambda| = 1$ and λ^* is also an eigenvalue of P.

Now let Q be a real orthogonal 3×3 matrix having $\lambda = 1$ as an eigenvalue of algebraic multiplicity 2. Give a geometrical description of the action of Q on $\mathbb{R}^3$, giving a reason for your answer. [You may assume that orthogonal matrices are always diagonalisable.]