UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Wednesday, 8 June, 2011 9:00 am to 12:00 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *. . .,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

(i) Find a primitive root modulo 17.

(ii) Let p be a prime of the form $2^m + 1$ for some integer $m \geqslant 1$. Prove that every quadratic non-residue modulo p is a primitive root modulo p.

2F Topics in Analysis

(i) Let $x_1, x_2, \ldots, x_n \in [-1, 1]$ be any set of n distinct numbers. Show that there exist numbers A_1, A_2, $\ldots$, A_n such that the formula

$$\int_{-1}^{1} p(x)\, dx = \sum_{j=1}^{n} A_j p(x_j)$$

is valid for every polynomial p of degree $\leqslant n - 1$.

(ii) For $n = 0, 1, 2, \ldots$, let p_n be the Legendre polynomial, over $[-1, 1]$, of degree n. Suppose that $x_1, x_2, \ldots, x_n \in [-1, 1]$ are the roots of p_n, and $A_1, A_2, \ldots, A_n$ are the numbers corresponding to $x_1, x_2, \ldots, x_n$ as in (i).

[You may assume without proof that for $n \geqslant 1$, p_n has n distinct roots in $[-1, 1]$.]

Prove that the integration formula in (i) is now valid for any polynomial p of degree $\leqslant 2n - 1$.

(iii) Is it possible to choose n distinct points $x_1, x_2, \ldots, x_n \in [-1, 1]$ and corresponding numbers $A_1, A_2, \ldots, A_n$ such that the integration formula in (i) is valid for any polynomial p of degree $\leqslant 2n$? Justify your answer.

3G Geometry and Groups

Let A and B be two rotations of the Euclidean plane $\mathbb{E}^2$ about centres a and b respectively. Show that the conjugate ABA^{-1} is also a rotation and find its fixed point. When do A and B commute? Show that the commutator $ABA^{-1}B^{-1}$ is a translation.

Deduce that any group of orientation-preserving isometries of the Euclidean plane either fixes a point or is infinite.

4G Coding and Cryptography

I happen to know that an apparently fair coin actually has probability p of heads with $1 > p > 1/2$. I play a very long sequence of games of heads and tails in which my opponent pays me back twice my stake if the coin comes down heads and takes my stake if the coin comes down tails. I decide to bet a proportion α of my fortune at the end of the nth game in the $(n+1)$st game. Determine, giving justification, the value α_0 maximizing the expected logarithm of my fortune in the long term, assuming I use the same α_0 at each game. Can it be actually disadvantageous for me to choose an $\alpha < \alpha_0$ (in the sense that I would be better off not playing)? Can it be actually disadvantageous for me to choose an $\alpha > \alpha_0$?

[Moral issues should be ignored.]

5J Statistical Modelling

Let f_0 be a probability density function, with cumulant generating function K. Define what it means for a random variable Y to have a model function of exponential dispersion family form, generated by f_0. Compute the cumulant generating function K_Y of Y and deduce expressions for the mean and variance of Y that depend only on first and second derivatives of K.

6B Mathematical Biology

A population with variable growth and harvesting is modelled by the equation

$$u_{t+1} = \max\left(\frac{ru_t^2}{1+u_t^2} - Eu_t, 0\right),$$

where r and E are positive constants.

Given that $r > 1$, show that a non-zero steady state exists if $0 < E < E_m(r)$, where $E_m(r)$ is to be determined.

Show using a cobweb diagram that, if $E < E_m(r)$, a non-zero steady state may be attained only if the initial population u_0 satisfies $\alpha < u_0 < \beta$, where α should be determined explicitly and β should be specified as a root of an algebraic equation.

With reference to the cobweb diagram, give an additional criterion that implies that $\alpha < u_0 < \beta$ is a sufficient condition, as well as a necessary condition, for convergence to a non-zero steady state.

7C Dynamical Systems

State the Poincaré–Bendixson theorem for two-dimensional dynamical systems.

A dynamical system can be written in polar coordinates (r, θ) as

$$\dot{r} = r - r^3(1 + \alpha \cos\theta)\,,$$
$$\dot{\theta} = 1 - r^2 \beta \cos\theta\,,$$

where α and β are constants with $0 < \alpha < 1$.

Show that trajectories enter the annulus $(1+\alpha)^{-1/2} < r < (1-\alpha)^{-1/2}$.

Show that if there is a fixed point (r_0, θ_0) inside the annulus then $r_0^2 = (\beta - \alpha)/\beta$ and $\cos\theta_0 = 1/(\beta - \alpha)$.

Use the Poincaré–Bendixson theorem to derive conditions on β that guarantee the existence of a periodic orbit.

8E Further Complex Methods

Find the two complex-valued functions $F^+(z)$ and $F^-(z)$ such that all of the following hold:

(i) $F^+(z)$ and $F^-(z)$ are analytic for $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ respectively, where $z = x + iy,\ x, y \in \mathbb{R}$.

(ii) $F^+(x) - F^-(x) = \frac{1}{x^4+1}\,, \quad x \in \mathbb{R}\,.$

(iii) $F^{\pm}(z) = O\left(\frac{1}{z}\right), \quad z \to \infty\,, \quad \operatorname{Im} z \neq 0\,.$

9C Classical Dynamics

Three particles, each of mass m, move along a straight line. Their positions on the line containing the origin, O, are x_1, x_2 and x_3. They are subject to forces derived from the potential energy function

$$V = \frac{1}{2} m\Omega^2 \left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 + x_1^2 + x_2^2 + x_3^2\right].$$

Obtain Lagrange's equations for the system, and show that the frequency, ω, of a normal mode satisfies

$$f^3 - 9f^2 + 24f - 16 = 0\,,$$

where $f = (\omega^2/\Omega^2)$. Find a complete set of normal modes for the system, and draw a diagram indicating the nature of the corresponding motions.

10E Cosmology

A spherically symmetric star in hydrostatic equilibrium has density $\rho(r)$ and pressure $P(r)$, which satisfy the pressure support equation,

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \tag{$*$}$$

where $m(r)$ is the mass within a radius r. Show that this implies

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\, r^2 \rho\,.$$

Provide a justification for choosing the boundary conditions $dP/dr = 0$ at the centre of the star ($r = 0$) and $P = 0$ at its outer radius ($r = R$).

Use the pressure support equation ($*$) to derive the virial theorem for a star,

$$\langle P \rangle V = -\frac{1}{3} E_{\text{grav}}\,,$$

where $\langle P \rangle$ is the average pressure, V is the total volume of the star and E_{grav} is its total gravitational potential energy.

SECTION II

11F Topics in Analysis

Let $C[0,1]$ be the space of real continuous functions on the interval $[0,1]$. A mapping $L : C[0,1] \to C[0,1]$ is said to be *positive* if $L(f) \geqslant 0$ for each $f \in C[0,1]$ with $f \geqslant 0$, and *linear* if $L(af+bg) = aL(f) + bL(g)$ for all functions $f, g \in C[0,1]$ and constants $a, b \in \mathbb{R}$.

(i) Let $L_n : C[0,1] \to C[0,1]$ be a sequence of positive, linear mappings such that $L_n(f) \to f$ uniformly on $[0,1]$ for the three functions $f(x) = 1, x, x^2$. Prove that $L_n(f) \to f$ uniformly on $[0,1]$ for every $f \in C[0,1]$.

(ii) Define $B_n : C[0,1] \to C[0,1]$ by

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}\,,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Using the result of part (i), or otherwise, prove that $B_n(f) \to f$ uniformly on $[0,1]$.

(iii) Let $f \in C[0,1]$ and suppose that

$$\int_0^1 f(x) x^{4n}\, dx = 0$$

for each $n = 0, 1, \ldots$. Prove that f must be the zero function.

[You should not use the Stone–Weierstrass theorem without proof.]

12G Coding and Cryptography

Define a cyclic code. Show that there is a bijection between the cyclic codes of length n and the factors of $X^n - 1$ over the field $\mathbb{F}_2$ of order 2.

What is meant by saying that α is a primitive nth root of unity in a finite field extension K of $\mathbb{F}_2$? What is meant by saying that C is a BCH code of length n with defining set $\{\alpha,\, \alpha^2, \ldots,\, \alpha^{\delta-1}\}$? Show that such a code has minimum distance at least δ.

Suppose that K is a finite field extension of $\mathbb{F}_2$ in which $X^7 - 1$ factorises into linear factors. Show that if β is a root of $X^3 + X^2 + 1$ then β is a primitive 7th root of unity and β^2 is also a root of $X^3 + X^2 + 1$. Quoting any further results that you need show that the BCH code of length 7 with defining set $\{\beta,\, \beta^2\}$ is the Hamming code.

[Results on the Vandermonde determinant may be used without proof provided they are quoted correctly.]

13B Mathematical Biology

Consider a population subject to the following birth–death process. When the number of individuals in the population is n, the probability of an increase from n to $n+1$ in unit time is $\beta n+\gamma$ and the probability of a decrease from n to $n-1$ is $\alpha n(n-1)$, where α, β and γ are constants.

Show that the master equation for $P(n,t)$, the probability that at time t the population has n members, is

$$\frac{\partial P}{\partial t} = \alpha n(n+1)P(n+1,t) - \alpha n(n-1)P(n,t) + (\beta n - \beta + \gamma)P(n-1,t) - (\beta n + \gamma)P(n,t)\,.$$

Show that $\langle n\rangle$, the mean number of individuals in the population, satisfies

$$\frac{d\langle n\rangle}{dt} = -\alpha\langle n^2\rangle + (\alpha+\beta)\langle n\rangle + \gamma\,.$$

Deduce that, in a steady state,

$$\langle n\rangle = \frac{\alpha+\beta}{2\alpha} \pm \sqrt{\frac{(\alpha+\beta)^2}{4\alpha^2} + \frac{\gamma}{\alpha} - (\Delta n)^2}\,,$$

where Δn is the standard deviation of n. When is the minus sign admissable?

Show how a Fokker–Planck equation of the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial n}[g(n)P(n,t)] + \frac{1}{2}\frac{\partial^2}{\partial n^2}[h(n)P(n,t)] \qquad (*)$$

may be derived under conditions to be explained, where the functions $g(n)$ and $h(n)$ should be evaluated.

In the case $\alpha \ll \gamma$ and $\beta = 0$, find the leading-order approximation to n_* such that $g(n_*) = 0$. Defining the new variable $x = n - n_*$, where $g(n_*) = 0$, approximate $g(n)$ by $g'(n_*)x$ and $h(n)$ by $h(n_*)$. Solve $(*)$ for $P(x)$ in the steady-state limit and deduce leading-order estimates for $\langle n\rangle$ and $(\Delta n)^2$.

14E Further Complex Methods

Consider the following sum related to Riemann's zeta function:

$$S := \sum_{m=1}^{\left[\frac{a}{2\pi}\right]} m^{s-1}, \quad s = \sigma + it, \ \sigma, t \in \mathbb{R}, \quad a > 2\pi, \ a \neq 2\pi N, \ N \in \mathbb{Z}^+,$$

where $[a/2\pi]$ denotes the integer part of $a/2\pi$.

(i) By using an appropriate branch cut, show that

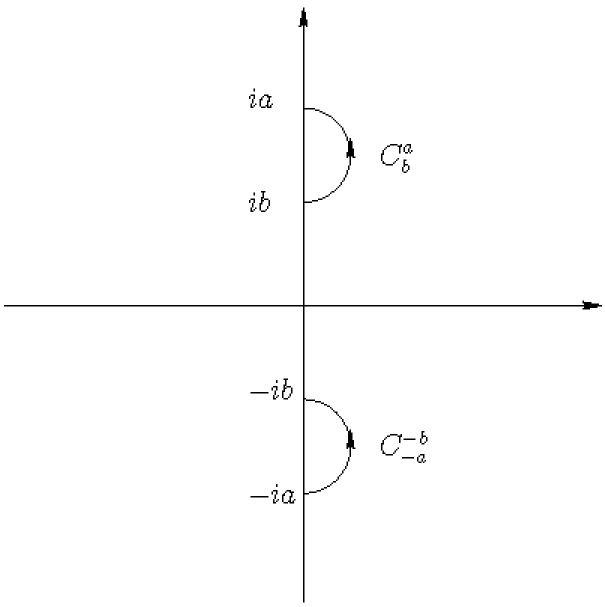
$$S = \frac{e^{-\frac{i\pi s}{2}}}{(2\pi)^s} \int_L f(z,s)\, dz\,, \qquad f(z,s) = \frac{e^{-z}}{1-e^{-z}} z^{s-1}\,,$$

where L is the circle in the complex z-plane centred at $i(a+b)/2$ with radius $(a-b)/2$, $0 < b < 2\pi$.

(ii) Use the above representation to show that, for $a > 2\pi$ and $0 < b < 2\pi$,

$$\sum_{m=1}^{\left[\frac{a}{2\pi}\right]} m^{s-1} = \frac{1}{(2\pi)^s} \left[e^{-\frac{i\pi s}{2}} \int_{C_b^a} f(z,s)\, dz - e^{\frac{i\pi s}{2}} \int_{C_{-a}^{-b}} f(z,s)\, dz + \frac{a^s}{s} - \frac{b^s}{s} \right],$$

where $f(z,s)$ is defined in (i) and the curves C_b^a, C_{-a}^{-b} are the following semi-circles in the right half complex z-plane:



The curves C_b^a and C_{-a}^{-b}.

$$C_b^a = \left\{ \frac{i(a+b)}{2} + \frac{(a-b)}{2} e^{i\theta}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right\},$$

$$C_{-a}^{-b} = \left\{ \frac{-i(a+b)}{2} + \frac{(a-b)}{2} e^{i\theta}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right\}.$$

15C Classical Dynamics

Derive Euler's equations governing the torque-free and force-free motion of a rigid body with principal moments of inertia I_1, I_2 and I_3, where $I_1 < I_2 < I_3$. Identify two constants of the motion. Hence, or otherwise, find the equilibrium configurations such that the angular-momentum vector, as measured with respect to axes fixed in the body, remains constant. Discuss the stability of these configurations.

A spacecraft may be regarded as moving in a torque-free and force-free environment. Nevertheless, flexing of various parts of the frame can cause significant dissipation of energy. How does the angular-momentum vector ultimately align itself within the body?

16H Logic and Set Theory

State and prove Zorn's Lemma. [You may assume Hartogs' Lemma.] Where in your argument have you made use of the Axiom of Choice?

Show that every real vector space has a basis.

Let V be a real vector space having a basis of cardinality $\aleph_1$. What is the cardinality of V? Justify your answer.

17F Graph Theory

What does it mean to say that a graph G is *k-colourable?* Define the *chromatic number* $\chi(G)$ of a graph G, and the *chromatic number* $\chi(S)$ of a closed surface S.

State the Euler–Poincaré formula relating the numbers of vertices, edges and faces in a drawing of a graph G on a closed surface S of Euler characteristic E. Show that if $E \leqslant 0$ then

$$\chi(S) \leqslant \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor .$$

Find, with justification, the chromatic number of the Klein bottle N_2. Show that if G is a triangle-free graph which can be drawn on the Klein bottle then $\chi(G) \leqslant 4$.

[You may assume that the Klein bottle has Euler characteristic 0, and that K_6 can be drawn on the Klein bottle but K_7 cannot. You may use Brooks's theorem.]

18H Galois Theory

Let $F = \mathbb{C}(x, y)$ be the function field in two variables x, y. Let $n \geqslant 1$, and $K = \mathbb{C}(x^n + y^n, xy)$ be the subfield of F of all rational functions in $x^n + y^n$ and xy.

(i) Let $K' = K(x^n)$, which is a subfield of F. Show that K'/K is a quadratic extension.

(ii) Show that F/K' is cyclic of order n, and F/K is Galois. Determine the Galois group $\mathrm{Gal}(F/K)$.

19I Representation Theory

State Maschke's Theorem for finite-dimensional complex representations of the finite group G. Show by means of an example that the requirement that G be finite is indispensable.

Now let G be a (possibly infinite) group and let H be a normal subgroup of finite index r in G. Let $g_1, \ldots, g_r$ be representatives of the cosets of H in G. Suppose that V is a finite-dimensional completely reducible $\mathbb{C}G$-module. Show that

(i) if U is a $\mathbb{C}H$-submodule of V and $g \in G$, then the set $gU = \{gu : u \in U\}$ is a $\mathbb{C}H$-submodule of V;

(ii) if U is a $\mathbb{C}H$-submodule of V, then $\sum_{i=1}^{r} g_i U$ is a $\mathbb{C}G$-submodule of V;

(iii) V is completely reducible regarded as a $\mathbb{C}H$-module.

Hence deduce that if χ is an irreducible character of the finite group G then all the constituents of χ_H have the same degree.

20F Number Fields

(i) Suppose that $d > 1$ is a square-free integer. Describe, with justification, the ring of integers in the field $K = \mathbb{Q}(\sqrt{d})$.

(ii) Show that $\mathbb{Q}(2^{1/3}) = \mathbb{Q}(4^{1/3})$ and that $\mathbb{Z}[4^{1/3}]$ is not the ring of integers in this field.

21H Algebraic Topology

Explain what is meant by a covering projection. State and prove the path-lifting property for covering projections, and indicate briefly how it generalizes to a lifting property for homotopies between paths. [You may assume the Lebesgue Covering Theorem.]

Let X be a simply connected space, and let G be a subgroup of the group of all homeomorphisms $X \to X$. Suppose that, for each $x \in X$, there exists an open neighbourhood U of x such that $U \cap g[U] = \emptyset$ for each $g \in G$ other than the identity. Show that the projection $p \colon X \to X/G$ is a covering projection, and deduce that $\Pi_1(X/G) \cong G$.

By regarding S^3 as the set of all quaternions of modulus 1, or otherwise, show that there is a quotient space of S^3 whose fundamental group is a non-abelian group of order 8.

22G Linear Analysis

State and prove the Baire Category Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $x \in \mathbb{R}$, define

$$\omega_f(x) = \inf_{\delta > 0} \sup_{\substack{|y-x| \leqslant \delta \\ |y'-x| \leqslant \delta}} |f(y) - f(y')| .$$

Show that f is continuous at x if and only if $\omega_f(x) = 0$.

Show that for any $\epsilon > 0$ the set $\{x \in \mathbb{R} : \omega_f(x) < \epsilon\}$ is open.

Hence show that the set of points at which f is continuous cannot be precisely the set $\mathbb{Q}$ of rationals.

23G Riemann Surfaces

Let Λ be a lattice in $\mathbb{C}$ generated by 1 and τ, where τ is a fixed complex number with non-zero imaginary part. Suppose that f is a meromorphic function on $\mathbb{C}$ for which the poles of f are precisely the points in Λ, and for which $f(z) - 1/z^2 \to 0$ as $z \to 0$. Assume moreover that $f'(z)$ determines a doubly periodic function with respect to Λ with $f'(-z) = -f'(z)$ for all $z \in \mathbb{C} \setminus \Lambda$. Prove that:

(i) $f(-z) = f(z)$ for all $z \in \mathbb{C} \setminus \Lambda$.

(ii) f is doubly periodic with respect to Λ.

(iii) If it exists, f is uniquely determined by the above properties.

(iv) For some complex number A, f satisfies the differential equation $f''(z) = 6f(z)^2 + A$.

24H Algebraic Geometry

(i) Let k be an algebraically closed field, and let I be an ideal in $k[x_0, \dots, x_n]$. Define what it means for I to be homogeneous.

Now let $Z \subseteq \mathbb{A}^{n+1}$ be a Zariski closed subvariety invariant under $k^* = k - \{0\}$; that is, if $z \in Z$ and $\lambda \in k^*$, then $\lambda z \in Z$. Show that $I(Z)$ is a homogeneous ideal.

(ii) Let $f \in k[x_1, \dots, x_{n-1}]$, and let $\Gamma = \{(x, f(x)) \mid x \in \mathbb{A}^{n-1}\} \subseteq \mathbb{A}^n$ be the graph of f. Let $\overline{\Gamma}$ be the closure of Γ in $\mathbb{P}^n$.

Write, in terms of f, the homogeneous equations defining $\overline{\Gamma}$.

Assume that k is an algebraically closed field of characteristic zero. Now suppose $n = 3$ and $f(x, y) = y^3 - x^2 \in k[x, y]$. Find the singular points of the projective surface $\overline{\Gamma}$.

25I Differential Geometry

Let $\alpha : I \to \mathbb{R}^3$ be a smooth curve parametrized by arc-length, with $\alpha''(s) \neq 0$ for all $s \in I$. Define what is meant by the *Frenet frame* $t(s), n(s), b(s)$, the *curvature* and *torsion* of α. State and prove the Frenet formulae.

By considering $\langle \alpha, t \times n \rangle$, or otherwise, show that, if for each $s \in I$ the vectors $\alpha(s)$, $t(s)$ and $n(s)$ are linearly dependent, then $\alpha(s)$ is a plane curve.

State and prove the isoperimetric inequality for C^1 regular plane curves.

[You may assume Wirtinger's inequality, provided you state it accurately.]

26K Probability and Measure

(i) Define the notions of a π-system and a d-system. State and prove Dynkin's lemma.

(ii) Let $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ denote two finite measure spaces. Define the σ-algebra $\mathcal{E}_1 \otimes \mathcal{E}_2$ and the product measure $\mu_1 \otimes \mu_2$. [You do not need to verify that such a measure exists.] State (without proof) Fubini's Theorem.

(iii) Let $(E, \mathcal{E}, \mu)$ be a measure space, and let f be a non-negative Borel-measurable function. Let G be the subset of $E \times \mathbb{R}$ defined by

$$G = \{(x, y) \in E \times \mathbb{R} : 0 \leqslant y \leqslant f(x)\}\,.$$

Show that $G \in \mathcal{E} \otimes \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$. Show further that

$$\int f \, d\mu = (\mu \otimes \lambda)(G)\,,$$

where λ is Lebesgue measure.

27J Applied Probability

(i) Explain briefly what is meant by saying that a continuous-time Markov chain $X(t)$ is a birth-and-death process with birth rates $\lambda_i > 0$, $i \geqslant 0$, and death rates $\mu_i > 0$, $i \geqslant 1$.

(ii) In the case where $X(t)$ is recurrent, find a sufficient condition on the birth and death parameters to ensure that

$$\lim_{t\to\infty} \mathbb{P}(X(t) = j) = \pi_j > 0, \quad j \geqslant 0,$$

and express π_j in terms of these parameters. State the reversibility property of $X(t)$.

Jobs arrive according to a Poisson process of rate $\lambda > 0$. They are processed individually, by a single server, the processing times being independent random variables, each with the exponential distribution of rate $\nu > 0$. After processing, the job either leaves the system, with probability p, $0 < p < 1$, or, with probability $1 - p$, it splits into two separate jobs which are both sent to join the queue for processing again. Let $X(t)$ denote the number of jobs in the system at time t.

(iii) In the case $1 + \lambda/\nu < 2p$, evaluate $\lim_{t\to\infty} \mathbb{P}(X(t) = j)$, $j = 0, 1, \ldots$, and find the expected time that the processor is busy between two successive idle periods.

(iv) What happens if $1 + \lambda/\nu \geqslant 2p$?

28K Principles of Statistics

Random variables $X_1, \ldots, X_n$ are independent and identically distributed from the normal distribution with unknown mean M and unknown precision (inverse variance) H. Show that the likelihood function, for data $X_1 = x_1, \ldots, X_n = x_n$, is

$$L_n(\mu, h) \propto h^{n/2} \exp\left(-\tfrac{1}{2}h\left\{n\left(\overline{x}-\mu\right)^2 + S\right\}\right),$$

where $\overline{x} := n^{-1}\sum_i x_i$ and $S := \sum_i (x_i - \overline{x})^2$.

A bivariate prior distribution for (M, H) is specified, in terms of hyperparameters $(\alpha_0, \beta_0, m_0, \lambda_0)$, as follows. The marginal distribution of H is $\Gamma(\alpha_0, \beta_0)$, with density

$$\pi(h) \propto h^{\alpha_0 - 1} e^{-\beta_0 h} \quad (h > 0),$$

and the conditional distribution of M, given $H = h$, is normal with mean m_0 and precision $\lambda_0 h$.

Show that the conditional prior distribution of H, given $\mathrm{M} = \mu$, is

$$H \mid \mathrm{M} = \mu \quad \sim \quad \Gamma\left(\alpha_0 + \tfrac{1}{2}, \beta_0 + \tfrac{1}{2}\lambda_0\left(\mu - m_0\right)^2\right).$$

Show that the posterior joint distribution of (M, H), given $X_1 = x_1, \ldots, X_n = x_n$, has the same form as the prior, with updated hyperparameters $(\alpha_n, \beta_n, m_n, \lambda_n)$ which you should express in terms of the prior hyperparameters and the data.

[You may use the identity

$$p(t-a)^2 + q(t-b)^2 = (t-\delta)^2 + pq(a-b)^2,$$

where $p + q = 1$ and $\delta = pa + qb$.]

Explain how you could implement Gibbs sampling to generate a random sample from the posterior joint distribution.

29K Optimization and Control

Consider an optimal stopping problem in which the optimality equation takes the form

$$F_t(x) = \max\{r(x), E[F_{t+1}(x_{t+1})]\}, \quad t = 1, \ldots, N-1,$$

$F_N(x) = r(x)$, and where $r(x) > 0$ for all x. Let S denote the stopping set of the *one-step-look-ahead rule*. Show that if S is closed (in a sense you should explain) then the one-step-look-ahead rule is optimal.

N biased coins are to be tossed successively. The probability that the ith coin toss will show a head is known to be p_i $(0 < p_i < 1)$. At most once, after observing a head, and before tossing the next coin, you may guess that you have just seen the last head (i.e. that all subsequent tosses will show tails). If your guess turns out to be correct then you win £1.

Suppose that you have not yet guessed 'last head', and the ith toss is a head. Show that it cannot be optimal to guess that this is the last head if

$$\frac{p_{i+1}}{q_{i+1}} + \cdots + \frac{p_N}{q_N} > 1,$$

where $q_j = 1 - p_j$.

Suppose that $p_i = 1/i$. Show that it is optimal to guess that the last head is the first head (if any) to occur after having tossed at least i^* coins, where $i^* \approx N/e$ when N is large.

30J Stochastic Financial Models

Consider a symmetric simple random walk $(Z_n)_{n\in\mathbb{Z}^+}$ taking values in statespace $I = h\mathbb{Z}^2 \equiv \{(ih, jh) : i, j \in \mathbb{Z}\}$, where $h \equiv N^{-1}$ (N an integer). Writing $Z_n \equiv (X_n, Y_n)$, the transition probabilities are given by

$$P(\Delta Z_n = (h,0)) = P(\Delta Z_n = (0,h)) = P(\Delta Z_n = (-h,0)) = P(\Delta Z_n = (0,-h)) = \frac{1}{4},$$

where $\Delta Z_n \equiv Z_n - Z_{n-1}$.

What does it mean to say that $(M_n, \mathcal{F}_n)_{n\in\mathbb{Z}^+}$ is a *martingale*? Find a condition on θ and λ such that

$$M_n = \exp(\theta X_n - \lambda Y_n)$$

is a martingale. If $\theta = i\alpha$ for some real α, show that M is a martingale if

$$e^{-\lambda h} = 2 - \cos(\alpha h) - \sqrt{(2-\cos(\alpha h))^2 - 1}\,. \qquad (*)$$

Suppose that the random walk Z starts at position $(0,1) \equiv (0, Nh)$ at time 0, and suppose that

$$\tau = \inf\{n : Y_n = 0\}\,.$$

Stating fully any results to which you appeal, prove that

$$E\exp(i\alpha X_\tau) = e^{-\lambda}\,,$$

where λ is as given at $(*)$. Deduce that as $N \to \infty$

$$E\exp(i\alpha X_\tau) \to e^{-|\alpha|}$$

and comment briefly on this result.

31A Partial Differential Equations

Consider the Schrödinger equation

$$i\partial_t\psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V(x)\psi(t,x) \ , \qquad x \in \mathbb{R}^n, \ t > 0 \ ,$$
$$\psi(t=0,x) = \psi_I(x) \ , \qquad x \in \mathbb{R}^n \ ,$$

where V is a smooth real-valued function.

Prove that, for smooth solutions, the following equations are valid for all $t > 0$:

(i)
$$\int_{\mathbb{R}^n} |\psi(t,x)|^2 \, dx = \int_{\mathbb{R}^n} |\psi_I(x)|^2 \, dx \ .$$

(ii)
$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla\psi(t,x)|^2 \, dx + \int_{\mathbb{R}^n} V(x)|\psi(t,x)|^2 \, dx$$
$$= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla\psi_I(x)|^2 \, dx + \int_{\mathbb{R}^n} V(x)|\psi_I(x)|^2 \, dx \ .$$

32A Integrable Systems

Consider the Poisson structure

$$\{F,G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)}\, dx\,, \tag{1}$$

where F, G are polynomial functionals of $u, u_x, u_{xx}, \ldots$. Assume that $u, u_x, u_{xx}, \ldots$ tend to zero as $|x| \to \infty$.

(i) Show that $\{F,G\} = -\{G,F\}$.

(ii) Write down Hamilton's equations for $u = u(x,t)$ corresponding to the following Hamiltonians:

$$H_0[u] = \int_{\mathbb{R}} \frac{1}{2} u^2\, dx\,, \quad H[u] = \int_{\mathbb{R}} \left(\frac{1}{2} {u_x}^2 + u^3 + u u_x \right) dx\,.$$

(iii) Calculate the Poisson bracket $\{H_0, H\}$, and hence or otherwise deduce that the following overdetermined system of partial differential equations for $u = u(x, t_0, t)$ is compatible:

$$u_{t_0} = u_x\,, \tag{2}$$

$$u_t = 6uu_x - u_{xxx}\,. \tag{3}$$

[You may assume that the Jacobi identity holds for (1).]

(iv) Find a symmetry of (3) generated by $X = \partial/\partial u + \alpha t \partial/\partial x$ for some constant $\alpha \in \mathbb{R}$ which should be determined. Construct a vector field Y corresponding to the one–parameter group

$$x \to \beta x\,, \quad t \to \gamma t\,, \quad u \to \delta u\,,$$

where (β, γ, δ) should be determined from the symmetry requirement. Find the Lie algebra generated by the vector fields (X, Y).

33D Principles of Quantum Mechanics

A quantum system has energy eigenstates $|n\rangle$ with eigenvalues $E_n = n\hbar$, $n \in \{1, 2, 3, \ldots\}$. An observable Q is such that $Q|n\rangle = q_n|n\rangle$.

(a) What is the commutator of Q with the Hamiltonian H?

(b) Given $q_n = \frac{1}{n}$, consider the state

$$|\psi\rangle \propto \sum_{n=1}^{N} \sqrt{n}|n\rangle\,.$$

Determine:

(i) The probability of measuring Q to be $1/N$.

(ii) The probability of measuring energy $\hbar$ followed by another immediate measurement of energy $2\hbar$.

(iii) The average of many separate measurements of Q, each measurement being on a state $|\psi\rangle$, as $N \to \infty$.

(c) Given $q_1 = 1$ and $q_n = -1$ for $n > 1$, consider the state

$$|\psi\rangle \propto \sum_{n=1}^{\infty} \alpha^{n/2}|n\rangle\,,$$

where $0 < \alpha < 1$.

(i) Show that the probability of measuring an eigenvalue $q = -1$ of $|\psi\rangle$ is

$$A + B\alpha\,,$$

where A and B are integers that you should find.

(ii) Show that $\langle Q\rangle_\psi$ is $C + D\alpha$, where C and D are integers that you should find.

(iii) Given that Q is measured to be -1 at time $t = 0$, write down the state after a time t has passed. What is then the subsequent probability at time t of measuring the energy to be $2\hbar$?

34E Applications of Quantum Mechanics

A beam of particles of mass m and momentum $p = \hbar k$, incident along the z-axis, is scattered by a spherically symmetric potential $V(r)$, where $V(r) = 0$ for large r. State the boundary conditions on the wavefunction as $r \to \infty$ and hence define the scattering amplitude $f(\theta)$, where θ is the scattering angle.

Given that, for large r,

$$e^{ikr\cos\theta} = \frac{1}{2ikr} \sum_{l=0}^{\infty} (2l+1) \left(e^{ikr} - (-1)^l e^{-ikr} \right) P_l(\cos\theta) \,,$$

explain how the partial-wave expansion can be used to define the phase shifts $\delta_l(k)$ $(l = 0, 1, 2, \ldots)$. Furthermore, given that $d\sigma/d\Omega = |f(\theta)|^2$, derive expressions for $f(\theta)$ and the total cross-section σ in terms of the δ_l.

In a particular case $V(r)$ is given by

$$V(r) = \begin{cases} \infty \,, & r < a \,, \\ -V_0 \,, & a < r < 2a \,, \\ 0 \,, & r > 2a \,, \end{cases}$$

where $V_0 > 0$. Show that the S-wave phase shift δ_0 satisfies

$$\tan(\delta_0) = \frac{k\cos(2ka) - \kappa\cot(\kappa a)\sin(2ka)}{k\sin(2ka) + \kappa\cot(\kappa a)\cos(2ka)} \,,$$

where $\kappa^2 = 2mV_0/\hbar^2 + k^2$.

Derive an expression for the scattering length a_s in terms of κ. Find the values of κ for which $|a_s|$ diverges and briefly explain their physical significance.

35D Statistical Physics

Write down the partition function for a single classical non-relativistic particle of mass m moving in three dimensions in a potential $U(\mathbf{x})$ and in equilibrium with a heat bath at temperature T.

A system of N non-interacting classical non-relativistic particles, in equilibrium at temperature T, is placed in a potential

$$U(\mathbf{x}) = \frac{(x^2+y^2+z^2)^n}{V^{2n/3}}\,,$$

where n is a positive integer. Using the partition function, show that the free energy is

$$F = -Nk_BT\left(\log V + \frac{3}{2}\frac{n+1}{n}\log k_BT + \log I_n + \text{const}\right)\,, \qquad (*)$$

where

$$I_n = \left(\frac{m}{2\pi\hbar^2}\right)^{3/2}\int_0^\infty 4\pi u^2 \exp(-u^{2n})\,du\,.$$

Explain the physical relevance of the constant term in the expression $(*)$.

Viewing V as an external parameter, akin to volume, compute the conjugate pressure p and show that the equation of state coincides with that of an ideal gas.

Compute the energy E, heat capacity C_V and entropy S of the gas. Determine the local particle number density as a function of $|\mathbf{x}|$.

36D General Relativity

The curvature tensor ${R^a}_{bcd}$ satisfies

$$V_{a;bc} - V_{a;cb} = V_e {R^e}_{abc}$$

for any covariant vector field V_a. Hence express ${R^e}_{abc}$ in terms of the Christoffel symbols and their derivatives. Show that

$${R^e}_{abc} = -{R^e}_{acb} \, .$$

Further, by setting $V_a = \partial\phi/\partial x^a$, deduce that

$${R^e}_{abc} + {R^e}_{cab} + {R^e}_{bca} = 0 \, .$$

Using local inertial coordinates or otherwise, obtain the Bianchi identities.

Define the Ricci tensor in terms of the curvature tensor and show that it is symmetric. [You may assume that $R_{abcd} = -R_{bacd}$.] Write down the contracted Bianchi identities.

In certain spacetimes of dimension $n \geqslant 2$, R_{abcd} takes the form

$$R_{abcd} = K(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}) \, .$$

Obtain the Ricci tensor and curvature scalar. Deduce, under some restriction on n which should be stated, that K is a constant.

37B Fluid Dynamics II

The energy equation for the motion of a viscous, incompressible fluid states that

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho u^2 \, dV + \int_S \tfrac{1}{2}\rho u^2 u_i n_i \, dS = \int_S u_i \sigma_{ij} n_j \, dS - 2\mu \int_V e_{ij} e_{ij} \, dV \, .$$

Interpret each term in this equation and explain the meaning of the symbols used.

Consider steady rectilinear flow in a (not necessarily circular) pipe having rigid stationary walls. Deduce a relation between the viscous dissipation per unit length of the pipe, the pressure gradient G, and the volume flux Q.

Starting from the Navier–Stokes equations, calculate the velocity field for steady rectilinear flow in a circular pipe of radius a. Using the relationship derived above, or otherwise, find the viscous dissipation per unit length of this flow in terms of G.

[*Hint: In cylindrical polar coordinates,*

$$\nabla^2 w(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) . \Big]$$

38B Waves

A uniform elastic solid with wavespeeds c_P and c_S occupies the region $z < 0$. An S-wave with displacement

$$\mathbf{u} = (\cos\theta, 0, -\sin\theta)\, \mathrm{e}^{\mathrm{i}k(x\sin\theta + z\cos\theta) - \mathrm{i}\omega t}$$

is incident from $z < 0$ on a rigid boundary at $z = 0$. Find the form and amplitudes of the reflected waves.

When is the reflected P-wave evanescent? Show that if the P-wave is evanescent then the amplitude of the reflected S-wave has the same magnitude as the incident wave, and interpret this result physically.

39A Numerical Analysis

Let $A \in \mathbb{R}^{n\times n}$ be a real matrix with n linearly independent eigenvectors. The eigenvalues of A can be calculated from the sequence $x^{(k)}$, $k = 0, 1, \ldots$, which is generated by the power method

$$x^{(k+1)} = \frac{Ax^{(k)}}{\|Ax^{(k)}\|},$$

where $x^{(0)}$ is a real nonzero vector.

(i) Describe the asymptotic properties of the sequence $x^{(k)}$ in the case that the eigenvalues λ_i of A satisfy $|\lambda_i| < |\lambda_n|$, $i = 1, \ldots, n-1$, and the eigenvectors are of unit length.

(ii) Present the implementation details for the power method for the setting in (i) and define the Rayleigh quotient.

(iii) Let A be the 3×3 matrix

$$A = \lambda I + P\,, \quad P = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where λ is real and nonzero. Find an explicit expression for A^k, $k = 1, 2, 3, \ldots$.

Let the sequence $x^{(k)}$ be generated by the power method as above. Deduce from your expression for A^k that the first and second components of $x^{(k+1)}$ tend to zero as $k \to \infty$. Further show that this implies $Ax^{(k+1)} - \lambda x^{(k+1)} \to 0$ as $k \to \infty$.

END OF PAPER