UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday, 6 June, 2011 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A****,* ***B****,* ***C****, . . . ,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Prove that, under the action of $\mathrm{SL}_2(\mathbb{Z})$, every positive definite binary quadratic form of discriminant -163, with integer coefficients, is equivalent to

$$x^2 + xy + 41y^2 .$$

2F Topics in Analysis

(i) State the Baire Category Theorem for metric spaces in its closed sets version.

(ii) Let $f : \mathbb{C} \to \mathbb{C}$ be a complex analytic function which is not a polynomial. Prove that there exists a point $z_0 \in \mathbb{C}$ such that each coefficient of the Taylor series of f at z_0 is non-zero.

3G Geometry and Groups

Let G be a finite subgroup of $\mathrm{SO}(3)$ and let Ω be the set of unit vectors that are fixed by some non-identity element of G. Show that the group G permutes the unit vectors in Ω and that Ω has at most three orbits. Describe these orbits when G is the group of orientation-preserving symmetries of a regular dodecahedron.

4G Coding and Cryptography

I think of an integer n with $1 \leqslant n \leqslant 10^6$. Explain how to find n using twenty questions (or less) of the form 'Is it true that $n \geqslant m$?' to which I answer yes or no.

I have watched a horse race with 15 horses. Is it possible to discover the order in which the horses finished by asking me twenty questions to which I answer yes or no?

Roughly how many questions of the yes/no type are required to discover the order in which n horses finished if n is large?

[You may assume that I answer honestly.]

5J Statistical Modelling

Let $Y_1, \ldots, Y_n$ be independent identically distributed random variables with model function $f(y, \theta)$, $y \in \mathcal{Y}$, $\theta \in \Theta \subseteq \mathbb{R}$, and denote by E_θ and Var_θ expectation and variance under $f(y, \theta)$, respectively. Define $U_n(\theta) = \sum_{i=1}^n \frac{\partial}{\partial \theta} \log f(Y_i, \theta)$. Prove that $E_\theta U_n(\theta) = 0$. Show moreover that if $T = T(Y_1, \ldots, Y_n)$ is any unbiased estimator of θ, then its variance satisfies $\mathrm{Var}_\theta(T) \geqslant (n\mathrm{Var}_\theta(U_1(\theta))^{-1}$. [You may use the Cauchy–Schwarz inequality without proof, and you may interchange differentiation and integration without justification if necessary.]

6B Mathematical Biology

A proposed model of insect dispersal is given by the equation

$$\frac{\partial n}{\partial t} = D \frac{\partial}{\partial x} \left[\left(\frac{n_0}{n} \right) \frac{\partial n}{\partial x} \right], \qquad (1)$$

where $n(x, t)$ is the density of insects and D and n_0 are constants.

Interpret the term on the right-hand side.

Explain why a solution of the form

$$n(x, t) = n_0 (Dt)^{-\beta} g(x/(Dt)^\beta), \qquad (2)$$

where β is a positive constant, can potentially represent the dispersal of a fixed number n_0 of insects initially localised at the origin.

Show that the equation (1) can be satisfied by a solution of the form (2) if $\beta = 1$ and find the corresponding function g.

7C Dynamical Systems

Find the fixed points of the dynamical system (with $\mu \neq 0$)

$$\dot{x} = \mu^2 x - xy\,,$$
$$\dot{y} = -y + x^2\,,$$

and determine their type as a function of μ.

Find the stable and unstable manifolds of the origin correct to order 4.

8E Further Complex Methods

Show that the following integral is well defined:

$$I(a,b) = \int_0^\infty \left(\frac{e^{-bx}}{e^{ia}e^x - 1} - \frac{e^{bx}}{e^{-ia}e^x - 1} \right) dx, \quad 0 < a < \infty,\ a \neq 2n\pi,\ n \in \mathbb{Z},\ 0 < b < 1\,.$$

Express $I(a,b)$ in terms of a combination of hypergeometric functions.

[You may assume without proof that the hypergeometric function $F(a,b;c;z)$ can be expressed in the form

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt\,,$$

for appropriate restrictions on c, b, z. Furthermore,

$$\Gamma(z+1) = z\Gamma(z)\,.]$$

9C Classical Dynamics

(i) A particle of mass m and charge q, at position $\mathbf{x}$, moves in an electromagnetic field with scalar potential $\phi(\mathbf{x},t)$ and vector potential $\mathbf{A}(\mathbf{x},t)$. Verify that the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - q(\phi - \dot{\mathbf{x}} \cdot \mathbf{A})$$

gives the correct equations of motion.

[Note that $\mathbf{E} = -\boldsymbol{\nabla}\phi - \dot{\mathbf{A}}$ and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$.]

(ii) Consider the case of a constant uniform magnetic field, with $\mathbf{E} = \mathbf{0}$, given by $\phi = 0$, $\mathbf{A} = (0, xB, 0)$, where (x,y,z) are Cartesian coordinates and B is a constant. Find the motion of the particle, and describe it carefully.

10E Cosmology

Light of wavelength λ_e emitted by a distant object is observed by us to have wavelength λ_0. The redshift z of the object is defined by

$$1+z=\frac{\lambda_0}{\lambda_e}\,.$$

Assuming that the object is at a fixed comoving distance from us in a homogeneous and isotropic universe with scale factor $a(t)$, show that

$$1+z=\frac{a(t_0)}{a(t_e)}\,,$$

where t_e is the time of emission and t_0 the time of observation (i.e. today).

[You may assume the non-relativistic Doppler shift formula $\Delta\lambda/\lambda=(v/c)\cos\theta$ for the shift $\Delta\lambda$ in the wavelength of light emitted by a nearby object travelling with velocity v at angle θ to the line of sight.]

Given that the object radiates energy L per unit time, explain why the rate at which energy passes through a sphere centred on the object and intersecting the Earth is $L/(1+z)^2$.

SECTION II

11G Geometry and Groups

Prove that a group of Möbius transformations is discrete if, and only if, it acts discontinuously on hyperbolic 3-space.

Let G be the set of Möbius transformations $z \mapsto \dfrac{az+b}{cz+d}$ with

$$a, b, c, d \in \mathbb{Z}[i] = \{u + iv : u, v \in \mathbb{Z}\} \qquad \text{and} \qquad ad - bc = 1 \ .$$

Show that G is a group and that it acts discontinuously on hyperbolic 3-space. Show that G contains transformations that are elliptic, parabolic, hyperbolic and loxodromic.

12G Coding and Cryptography

Describe the Rabin–Williams coding scheme. Show that any method for breaking it will enable us to factorise the product of two primes.

Explain how the Rabin–Williams scheme can be used for bit sharing (that is to say 'tossing coins by phone').

13J Statistical Modelling

The data consist of the record times in 1984 for 35 Scottish hill races. The columns list the record time in minutes, the distance in miles, and the total height gained during the route. The data are displayed in R as follows (abbreviated):

```
> hills
                 dist climb    time
Greenmantle       2.5   650  16.083
Carnethy          6.0  2500  48.350
Craig Dunain      6.0   900  33.650
Ben Rha           7.5   800  45.600
Ben Lomond        8.0  3070  62.267
[...]
Cockleroi         4.5   850  28.100
Moffat Chase     20.0  5000 159.833
```

Consider a simple linear regression of `time` on `dist` and `climb`. Write down this model mathematically, and explain any assumptions that you make. How would you instruct R to fit this model and assign it to a variable `hills.lm1`?

First, we test the hypothesis of no linear relationship to the variables `dist` and `climb` against the full model. R provides the following ANOVA summary:

```
  Res.Df   RSS Df Sum of Sq      F    Pr(>F)
1     34 85138
2     32  6892  2     78247 181.66 < 2.2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Using the information in this table, explain carefully how you would test this hypothesis. What do you conclude?

The R command

```
summary(hills.lm1)
```

provides the following (slightly abbreviated) summary:

```
[...]
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -8.992039   4.302734  -2.090   0.0447 *
dist         6.217956   0.601148  10.343 9.86e-12 ***
climb        0.011048   0.002051   5.387 6.45e-06 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
[...]
```

Carefully explain the information that appears in each column of the table. What are your conclusions? In particular, how would you test for the significance of the variable `climb` in this model?

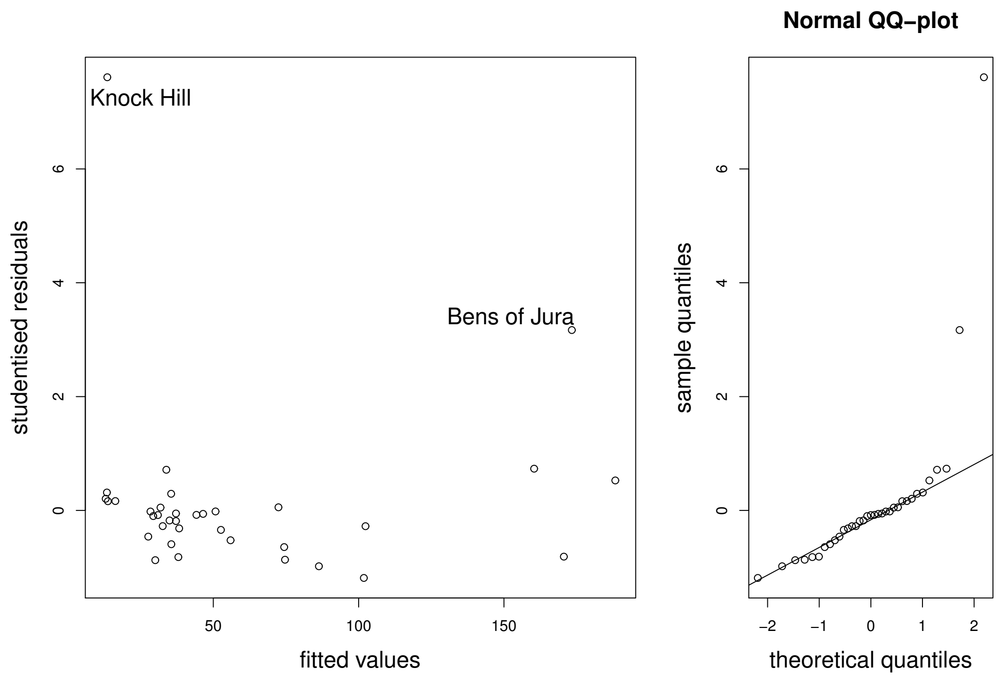


Figure 1: Hills data: diagnostic plots

Finally, we perform model diagnostics on the full model, by looking at studentised residuals versus fitted values, and the normal QQ-plot. The plots are displayed in Figure 1. Comment on possible sources of model misspecification. Is it possible that the problem lies with the data? If so, what do you suggest?

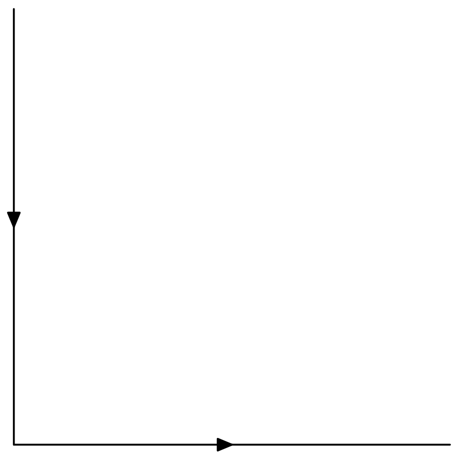
14E Further Complex Methods

(i) By assuming the validity of the Fourier transform pair, prove the validity of the following transform pair:

$$\hat{q}(k) = \int_0^\infty e^{-ikx} q(x)\, dx\,, \quad \operatorname{Im} k \leqslant 0\,, \tag{1a}$$

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{q}(k)\, dk + \frac{c}{2\pi}\int_L e^{ikx}\hat{q}(-k)\, dk\,, \quad 0 < x < \infty\,, \tag{1b}$$

where c is an arbitrary complex constant and L is the union of the two rays $\arg k = \frac{\pi}{2}$ and $\arg k = 0$ with the orientation shown in the figure below:



The contour L.

(ii) Verify that the partial differential equation

$$iq_t + q_{xx} = 0, \quad 0 < x < \infty\,,\ t > 0\,, \tag{2}$$

can be rewritten in the following form:

$$\left(e^{-ikx+ik^2t}q\right)_t - \left[e^{-ikx+ik^2t}\left(-kq + iq_x\right)\right]_x = 0\,, \quad k \in \mathbb{C}\,. \tag{3}$$

Consider equation (2) supplemented with the conditions

$$\begin{aligned} &q(x,0) = q_0(x)\,, \quad 0 < x < \infty\,,\\ &q(x,t) \text{ vanishes sufficiently fast for all } t \text{ as } x \to \infty\,. \end{aligned} \tag{4}$$

By using equations (1a) and (3), show that

$$\hat{q}(k,t) = e^{-ik^2t}\hat{q}_0(k) + e^{-ik^2t}\left[k\tilde{g}_0(k^2,t) - i\tilde{g}_1(k^2,t)\right]\,,\ \operatorname{Im} k \leqslant 0\,, \tag{5}$$

where

$$\hat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x)\, dx\,, \quad \operatorname{Im} k \leqslant 0\,,$$

$$\tilde{g}_0(k,t) = \int_0^t e^{ik\tau} q(0,\tau)\, d\tau\,, \quad \tilde{g}_1(k,t) = \int_0^t e^{ik\tau} q_x(0,\tau)\, d\tau\,,\ k \in \mathbb{C}\,,\ t > 0\,.$$

Use (1b) to invert equation (5) and furthermore show that

$$\int_{-\infty}^{\infty} e^{ikx-ik^2t}\left[k\tilde{g}_0(k^2,t) + i\tilde{g}_1(k^2,t)\right] dk = \int_L e^{ikx-ik^2t}\left[k\tilde{g}_0(k^2,t) + i\tilde{g}_1(k^2,t)\right] dk,\ t > 0\,, x > 0\,.$$

Hence determine the constant c so that the solution of equation (2), with the conditions (4) and with the condition that either $q(0,t)$ or $q_x(0,t)$ is given, can be expressed in terms of an integral involving $\hat{q}_0(k)$ and either $\tilde{g}_0$ or $\tilde{g}_1$.

15E Cosmology

A homogeneous and isotropic universe, with scale factor a, curvature parameter k, energy density ρ and pressure P, satisfies the Friedmann and energy conservation equations

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho\,,$$
$$\dot{\rho} + 3H(\rho + P/c^2) = 0\,,$$

where $H = \dot{a}/a$, and the dot indicates a derivative with respect to cosmological time t.

(i) Derive the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2)\,.$$

Given that the strong energy condition $\rho c^2 + 3P \geqslant 0$ is satisfied, show that $(aH)^2$ is a decreasing function of t in an expanding universe. Show also that the density parameter $\Omega = 8\pi G\rho/(3H^2)$ satisfies

$$\Omega - 1 = \frac{kc^2}{a^2H^2}\,.$$

Hence explain, briefly, the flatness problem of standard big bang cosmology.

(ii) A flat ($k = 0$) homogeneous and isotropic universe is filled with a radiation fluid ($w_R = 1/3$) and a dark energy fluid ($w_\Lambda = -1$), each with an equation of state of the form $P_i = w_i\rho_i c^2$ and density parameters today equal to Ω_{R0} and $\Omega_{\Lambda 0}$ respectively. Given that each fluid independently obeys the energy conservation equation, show that the total energy density $(\rho_R + \rho_\Lambda)c^2$ equals ρc^2, where

$$\rho(t) = \frac{3H_0^2}{8\pi G}\,\frac{\Omega_{R0}}{a^4}\left(1 \;+\; \frac{1-\Omega_{R0}}{\Omega_{R0}}\,a^4\right)\,,$$

with H_0 being the value of the Hubble parameter today. Hence solve the Friedmann equation to get

$$a(t) = \alpha(\sinh\beta t)^{1/2}\,,$$

where α and β should be expressed in terms Ω_{R0} and $\Omega_{\Lambda 0}$. Show that this result agrees with the expected asymptotic solutions at both early ($t \to 0$) and late ($t \to \infty$) times.

[*Hint:* $\int dx/\sqrt{x^2+1} = \operatorname{arcsinh} x$.]

16H Logic and Set Theory

Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent.

Which of the following assertions about ordinals α, β and γ are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) $\alpha\beta = \beta\alpha$.

(ii) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

(iii) If $\alpha \geqslant \omega^2$ then $\alpha + \omega^2 = \omega^2 + \alpha$.

(iv) If $\alpha \geqslant \omega_1$ then $\alpha\omega_1 = \omega_1\alpha$.

17F Graph Theory

Let G be a bipartite graph with vertex classes X and Y. What is a *matching* from X to Y?

Show that if $|\Gamma(A)| \geqslant |A|$ for all $A \subset X$ then G contains a matching from X to Y.

Let d be a positive integer. Show that if $|\Gamma(A)| \geqslant |A| - d$ for all $A \subset X$ then G contains a set of $|X| - d$ independent edges.

Show that if 0 is not an eigenvalue of G then G contains a matching from X to Y.

Suppose now that $|X| = |Y| \geqslant 1$ and that G does contain a matching from X to Y. Must it be the case that 0 is not an eigenvalue of G? Justify your answer.

18H Galois Theory

Let K be a field.

(i) Let F and F' be two finite extensions of K. When the degrees of these two extensions are equal, show that every K-homomorphism $F \to F'$ is an isomorphism. Give an example, with justification, of two finite extensions F and F' of K, which have the same degrees but are not isomorphic over K.

(ii) Let L be a finite extension of K. Let F and F' be two finite extensions of L. Show that if F and F' are isomorphic as extensions of L then they are isomorphic as extensions of K. Prove or disprove the converse.

19I Representation Theory

Let G be a finite group and Z its centre. Suppose that G has order n and Z has order m. Suppose that $\rho : G \to \mathrm{GL}(V)$ is a complex irreducible representation of degree d.

(i) For $g \in Z$, show that $\rho(g)$ is a scalar multiple of the identity.

(ii) Deduce that $d^2 \leqslant n/m$.

(iii) Show that, if ρ is faithful, then Z is cyclic.

[Standard results may be quoted without proof, provided they are stated clearly.]

Now let G be a group of order 18 containing an elementary abelian subgroup P of order 9 and an element t of order 2 with $txt^{-1} = x^{-1}$ for each $x \in P$. By considering the action of P on an irreducible $\mathbb{C}G$-module prove that G has no faithful irreducible complex representation.

20F Number Fields

Calculate the class group for the field $K = \mathbb{Q}(\sqrt{-17})$.

[You may use any general theorem, provided that you state it accurately.]

Find all solutions in $\mathbb{Z}$ of the equation $y^2 = x^5 - 17$.

21H Algebraic Topology

Are the following statements true or false? Justify your answers.

(i) If x and y lie in the same path-component of X, then $\Pi_1(X, x) \cong \Pi_1(X, y)$.

(ii) If x and y are two points of the Klein bottle K, and u and v are two paths from x to y, then u and v induce the same isomorphism from $\Pi_1(K, x)$ to $\Pi_1(K, y)$.

(iii) $\Pi_1(X \times Y, (x, y))$ is isomorphic to $\Pi_1(X, x) \times \Pi_1(Y, y)$ for any two spaces X and Y.

(iv) If X and Y are connected polyhedra and $H_1(X) \cong H_1(Y)$, then $\Pi_1(X) \cong \Pi_1(Y)$.

22G Linear Analysis

State a version of the Stone–Weierstrass Theorem for real-valued functions on a compact metric space.

Suppose that $K : [0,1]^2 \to \mathbb{R}$ is a continuous function. Show that $K(x,y)$ may be uniformly approximated by functions of the form $\sum_{i=1}^n f_i(x)g_i(y)$ with $f_i, g_i : [0,1] \to \mathbb{R}$ continuous.

Let X, Y be Banach spaces and suppose that $T : X \to Y$ is a bounded linear operator. What does it mean to say that T is finite-rank? What does it mean to say that T is compact? Give an example of a bounded linear operator from $C[0,1]$ to itself which is not compact.

Suppose that $(T_n)_{n=1}^\infty$ is a sequence of finite-rank operators and that $T_n \to T$ in the operator norm. Briefly explain why the T_n are compact. Show that T is compact.

Hence, show that the integral operator $T : C[0,1] \to C[0,1]$ defined by

$$Tf(x) = \int_0^1 f(y)K(x,y)\,dy$$

is compact.

23G Riemann Surfaces

Suppose that R_1 and R_2 are Riemann surfaces, and A is a discrete subset of R_1. For any continuous map $\alpha : R_1 \to R_2$ which restricts to an analytic map of Riemann surfaces $R_1 \setminus A \to R_2$, show that α is an analytic map.

Suppose that f is a non-constant analytic function on a Riemann surface R. Show that there is a discrete subset $A \subset R$ such that, for $P \in R \setminus A$, f defines a local chart on some neighbourhood of P.

Deduce that, if $\alpha : R_1 \to R_2$ is a homeomorphism of Riemann surfaces and f is a non-constant analytic function on R_2 for which the composite $f \circ \alpha$ is analytic on R_1, then α is a conformal equivalence. Give an example of a pair of Riemann surfaces which are homeomorphic but not conformally equivalent.

[You may assume standard results for analytic functions on domains in the complex plane.]

24H Algebraic Geometry

(i) Let X be an affine variety over an algebraically closed field. Define what it means for X to be *irreducible*, and show that if U is a non-empty open subset of an irreducible X, then U is dense in X.

(ii) Show that $n \times n$ matrices with distinct eigenvalues form an affine variety, and are a Zariski open subvariety of affine space $\mathbb{A}^{n^2}$ over an algebraically closed field.

(iii) Let $\mathrm{char}_A(x) = \det(xI - A)$ be the characteristic polynomial of A. Show that the $n \times n$ matrices A such that $\mathrm{char}_A(A) = 0$ form a Zariski closed subvariety of $\mathbb{A}^{n^2}$.

Hence conclude that this subvariety is all of $\mathbb{A}^{n^2}$.

25I Differential Geometry

Let X and Y be manifolds and $f : X \to Y$ a smooth map. Define the notions *critical point*, *critical value*, *regular value* of f. Prove that if y is a regular value of f, then $f^{-1}(y)$ (if non-empty) is a smooth manifold of dimension $\dim X - \dim Y$.

[The Inverse Function Theorem may be assumed without proof if accurately stated.]

Let $M_n(\mathbb{R})$ be the set of all real $n \times n$ matrices and $\mathrm{SO}(n) \subset M_n(\mathbb{R})$ the group of all orthogonal matrices with determinant 1. Show that $\mathrm{SO}(n)$ is a smooth manifold and find its dimension.

Show further that $\mathrm{SO}(n)$ is compact and that its tangent space at $A \in \mathrm{SO}(n)$ is given by all matrices H such that $AH^t + HA^t = 0$.

26K Probability and Measure

(i) Let $(E, \mathcal{E}, \mu)$ be a measure space and let $1 \leqslant p < \infty$. For a measurable function f, let $\|f\|_p = (\int |f|^p d\mu)^{1/p}$. Give the definition of the space L^p. Prove that $(L^p, \|\cdot\|_p)$ forms a Banach space.

[You may assume that L^p is a normed vector space. You may also use in your proof any other result from the course provided that it is clearly stated.]

(ii) Show that convergence in probability implies convergence in distribution.

[*Hint: Show the pointwise convergence of the characteristic function, using without proof the inequality* $|e^{iy} - e^{ix}| \leqslant |x - y|$ *for* $x, y \in \mathbb{R}$.]

(iii) Let $(\alpha_j)_{j \geqslant 1}$ be a given real-valued sequence such that $\sum_{j=1}^{\infty} \alpha_j^2 = \sigma^2 < \infty$. Let $(X_j)_{j \geqslant 1}$ be a sequence of independent standard Gaussian random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let

$$Y_n = \sum_{j=1}^{n} \alpha_j X_j \, .$$

Prove that there exists a random variable Y such that $Y_n \to Y$ in L^2.

(iv) Specify the distribution of the random variable Y defined in part (iii), justifying carefully your answer.

27J Applied Probability

(i) Let X be a Markov chain with finitely many states. Define a stopping time and state the strong Markov property.

(ii) Let X be a Markov chain with state-space $\{-1, 0, 1\}$ and Q-matrix

$$Q = \begin{pmatrix} -(q+\lambda) & \lambda & q \\ 0 & 0 & 0 \\ q & \lambda & -(q+\lambda) \end{pmatrix}, \quad \text{where } q, \lambda > 0\,.$$

Consider the integral $\int_0^t X(s)\mathrm{d}s$, the signed difference between the times spent by the chain at states $+1$ and -1 by time t, and let

$$\begin{aligned} Y &= \sup\left[\int_0^t X(s)\mathrm{d}s :\ t > 0\right], \\ \psi_{\pm}(c) &= \mathbb{P}\left(Y > c \,\middle|\, X_0 = \pm 1\right), \quad c > 0\,. \end{aligned}$$

Derive the equation

$$\psi_-(c) = \int_0^\infty q e^{-(\lambda+q)u} \psi_+(c+u)\,\mathrm{d}u\,.$$

(iii) Obtain another equation relating ψ_+ to ψ_-.

(iv) Assuming that $\psi_+(c) = e^{-cA}$, $c > 0$, where A is a non-negative constant, calculate A.

(v) Give an intuitive explanation why the function ψ_+ must have the exponential form $\psi_+(c) = e^{-cA}$ for some A.

28K Principles of Statistics

Define *admissible*, *Bayes*, *minimax* decision rules.

A random vector $X = (X_1, X_2, X_3)^{\mathrm{T}}$ has independent components, where X_i has the normal distribution $\mathcal{N}(\theta_i, 1)$ when the parameter vector Θ takes the value $\theta = (\theta_1, \theta_2, \theta_3)^{\mathrm{T}}$. It is required to estimate Θ by a point $a \in \mathbb{R}^3$, with loss function $L(\theta, a) = \|a - \theta\|^2$. What is the risk function of the maximum-likelihood estimator $\widehat{\Theta} := X$? Show that $\widehat{\Theta}$ is dominated by the estimator $\widetilde{\Theta} := (1 - \|X\|^{-2})\,X$.

29J Stochastic Financial Models

In a one-period market, there are n assets whose prices at time t are given by $S_t = (S_t^1, \dots, S_t^n)^T$, $t = 0, 1$. The prices S_1 of the assets at time 1 have a $N(\mu, V)$ distribution, with non-singular covariance V, and the prices S_0 at time 0 are known constants. In addition, there is a bank account giving interest r, so that one unit of cash invested at time 0 will be worth $(1+r)$ units of cash at time 1.

An agent with initial wealth w_0 chooses a portfolio $\theta = (\theta^1, \dots, \theta^n)$ of the assets to hold, leaving him with $x = w_0 - \theta \cdot S_0$ in the bank account. His objective is to maximize his expected utility

$$E\left(-\exp\left[\,-\gamma\,\{\,x(1+r) + \theta \cdot S_1\,\}\right]\right) \qquad (\gamma > 0)\,.$$

Find his optimal portfolio in each of the following three situations:

(i) θ is unrestricted;

(ii) no investment in the bank account is allowed: $x = 0$;

(iii) the initial holdings x of cash must be non-negative.

For the third problem, show that the optimal initial holdings of cash will be zero if and only if

$$\frac{S_0 \cdot (\gamma V)^{-1}\mu - w_0}{S_0 \cdot (\gamma V)^{-1} S_0} \geqslant 1 + r\,.$$

30A Partial Differential Equations

Let $H = H(x,v)$, $x, v \in \mathbb{R}^n$, be a smooth real-valued function which maps $\mathbb{R}^{2n}$ into $\mathbb{R}$. Consider the initial value problem for the equation

$$\begin{aligned} &f_t + \nabla_v H \cdot \nabla_x f - \nabla_x H \cdot \nabla_v f = 0, \quad x, v \in \mathbb{R}^n,\ t > 0\,, \\ &f(x,v,t=0) = f_I(x,v), \quad x, v \in \mathbb{R}^n\,, \end{aligned}$$

for the unknown function $f = f(x,v,t)$.

(i) Use the method of characteristics to solve the initial value problem, locally in time.

(ii) Let $f_I \geqslant 0$ on $\mathbb{R}^{2n}$. Use the method of characteristics to prove that f remains non-negative (as long as it exists).

(iii) Let $F : \mathbb{R} \to \mathbb{R}$ be smooth. Prove that

$$\int_{\mathbb{R}^{2n}} F(f(x,v,t))\, dx\, dv = \int_{\mathbb{R}^{2n}} F(f_I(x,v))\, dx\, dv\,,$$

as long as the solution exists.

(iv) Let H be independent of x, namely $H(x,v) = a(v)$, where a is smooth and real-valued. Give the explicit solution of the initial value problem.

31A Asymptotic Methods

A function $f(n)$, defined for positive integer n, has an asymptotic expansion for large n of the following form:

$$f(n) \sim \sum_{k=0}^{\infty} a_k \frac{1}{n^{2k}}\,, \qquad n \to \infty\,. \qquad (*)$$

What precisely does this mean?

Show that the integral

$$I(n) = \int_0^{2\pi} \frac{\cos nt}{1+t^2}\, dt$$

has an asymptotic expansion of the form $(*)$. [The Riemann–Lebesgue lemma may be used without proof.] Evaluate the coefficients a_0, a_1 and a_2.

32A Integrable Systems

Define a finite-dimensional integrable system and state the Arnold–Liouville theorem.

Consider a four-dimensional phase space with coordinates (q_1, q_2, p_1, p_2), where $q_2 > 0$ and q_1 is periodic with period 2π. Let the Hamiltonian be

$$H = \frac{(p_1)^2}{2(q_2)^2} + \frac{(p_2)^2}{2} - \frac{k}{q_2}, \quad \text{where } k > 0\,.$$

Show that the corresponding Hamilton equations form an integrable system.

Determine the sign of the constant E so that the motion is periodic on the surface $H = E$. Demonstrate that in this case, the action variables are given by

$$I_1 = p_1\,, \quad I_2 = \gamma \int_\alpha^\beta \frac{\sqrt{(q_2-\alpha)(\beta-q_2)}}{q_2}\, dq_2\,,$$

where α, β, γ are positive constants which you should determine.

33D Principles of Quantum Mechanics

Two individual angular momentum states $|j_1,\ m_1\rangle$, $|j_2,\ m_2\rangle$, acted on by $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ respectively, can be combined to form a combined state $|J,\ M\rangle$. What is the combined angular momentum operator $\mathbf{J}$ in terms of $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$? [Units in which $\hbar = 1$ are to be used throughout.]

Defining raising and lowering operators $J_{\pm}^{(i)}$, where $i \in \{1, 2\}$, find an expression for $\mathbf{J}^2$ in terms of $\mathbf{J}^{(i)^2}$, $J_{\pm}^{(i)}$ and $J_3^{(i)}$. Show that this implies

$$\left[\mathbf{J}^2,\ J_3\right] = 0\,.$$

Write down the state with $J = j_1 + j_2$ and with J_3 eigenvalue $M = -j_1 - j_2$ in terms of the individual angular momentum states. From this starting point, calculate the combined state with eigenvalues $J = j_1 + j_2 - 1$ and $M = -j_1 - j_2 + 1$ in terms of the individual angular momentum states.

If $j_1 = 3$ and $j_2 = 1$ and the combined system is in the state $|3,\ -3\rangle$, what is the probability of measuring the $J_3^{(i)}$ eigenvalues of individual angular momentum states to be -3 and 0, respectively?

[You may assume without proof that standard angular momentum states $|j,\ m\rangle$ are joint eigenstates of $\mathbf{J}^2$ and J_3, obeying

$$J_{\pm}|j,\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\ |j,\ m+1\rangle,$$

and that

$$[J_{\pm},\ J_3] = \pm J_{\pm}\,.\ ]$$

34E Applications of Quantum Mechanics

In one dimension a particle of mass m and momentum $\hbar k$, $k > 0$, is scattered by a potential $V(x)$ where $V(x) \to 0$ as $|x| \to \infty$. Incoming and outgoing plane waves of positive $(+)$ and negative $(-)$ parity are given, respectively, by

$$\begin{aligned} I_+(k,x) &= e^{-ik|x|} \,, \qquad & I_-(k,x) &= \operatorname{sgn}(x) e^{-ik|x|} \,, \\ O_+(k,x) &= e^{ik|x|} \,, \qquad & O_-(k,x) &= -\operatorname{sgn}(x) e^{ik|x|} \,. \end{aligned}$$

The scattering solutions to the time-independent Schrödinger equation with positive and negative parity incoming waves are $\psi_+(x)$ and $\psi_-(x)$, respectively. State how the asymptotic behaviour of ψ_+ and ψ_- can be expressed in terms of I_+, I_-, O_+, O_- and the S-matrix denoted by

$$\boldsymbol{S} = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} \,.$$

In the case where $V(x) = V(-x)$ explain briefly why you expect $S_{+-} = S_{-+} = 0$.

The potential $V(x)$ is given by

$$V(x) = V_0[\delta(x-a) + \delta(x+a)] \,,$$

where V_0 is a constant. In this case, show that

$$S_{--}(k) = e^{-2ika} \left[\frac{(2k - iU_0)e^{ika} + iU_0 e^{-ika}}{(2k + iU_0)e^{-ika} - iU_0 e^{ika}} \right] \,,$$

where $U_0 = 2mV_0/\hbar^2$. Verify that $|S_{--}|^2 = 1$ and explain briefly the physical meaning of this result.

For $V_0 < 0$, by considering the poles or zeros of $S_{--}(k)$ show that there exists one bound state of negative parity in this potential if $U_0 a < -1$.

For $V_0 > 0$ and $U_0 a \gg 1$, show that $S_{--}(k)$ has a pole at

$$ka = \pi + \alpha - i\gamma$$

where, to leading order in $1/(U_0 a)$,

$$\alpha = -\frac{\pi}{U_0 a} \,, \qquad \gamma = \left(\frac{\pi}{U_0 a} \right)^2 \,.$$

Explain briefy the physical meaning of this result, and why you expect that $\gamma > 0$.

35D Statistical Physics

Describe the physical relevance of the microcanonical, canonical and grand canonical ensembles. Explain briefly the circumstances under which all ensembles are equivalent.

The Gibbs entropy for a probability distribution $p(n)$ over states is

$$S = -k_B \sum_n p(n) \log p(n) \,.$$

By imposing suitable constraints on $p(n)$, show how maximising the entropy gives rise to the probability distributions for the microcanonical and canonical ensembles.

A system consists of N non-interacting particles fixed at points in a lattice. Each particle has three states with energies $E = -\epsilon, 0, +\epsilon$. If the system is at a fixed temperature T, determine the average energy E and the heat capacity C. Evaluate each in the limits $T \to \infty$ and $T \to 0$.

Describe a configuration of the system that would have negative temperature. Does this system obey the third law of thermodynamics?

36C Electrodynamics

In the Landau–Ginzburg model of superconductivity, the energy of the system is given, for constants α and β, by

$$E = \int \left\{ \frac{1}{2\mu_0}\mathbf{B}^2 + \frac{1}{2m}[(i\hbar\nabla - q\mathbf{A})\psi^* \cdot (-i\hbar\nabla - q\mathbf{A})\psi] + \alpha\psi^*\psi + \beta(\psi^*\psi)^2 \right\} d^3\mathbf{x} \,,$$

where $\mathbf{B}$ is the time-independent magnetic field derived from the vector potential $\mathbf{A}$, and ψ is the wavefunction of the charge carriers, which have mass m and charge q.

Describe the physical meaning of each of the terms in the integral.

Explain why in a superconductor one must choose $\alpha < 0$ and $\beta > 0$. Find an expression for the number density n of the charge carriers in terms of α and β.

Show that the energy is invariant under the gauge transformations

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda \,, \qquad \psi \to \psi\, e^{iq\Lambda/\hbar} \,.$$

Assuming that the number density n is uniform, show that, if E is a minimum under variations of $\mathbf{A}$, then

$$\operatorname{curl}\mathbf{B} = -\frac{\mu_0 q^2 n}{m}(\mathbf{A} - \frac{\hbar}{q}\nabla\phi) \,,$$

where $\phi = \arg\psi$.

Find a formula for $\nabla^2\mathbf{B}$ and use it to explain why there cannot be a magnetic field inside the bulk of a superconductor.

37D General Relativity

Consider a metric of the form

$$ds^2 = -2\,du\,dv + dx^2 + dy^2 - 2H(u,x,y)du^2\,.$$

Let $x^a(\lambda)$ describe an affinely-parametrised geodesic, where $x^a \equiv (x^1, x^2, x^3, x^4) = (u, v, x, y)$. Write down explicitly the Lagrangian

$$L = g_{ab}\dot{x}^a\dot{x}^b\,,$$

with $\dot{x}^a = dx^a/d\lambda$, using the given metric. Hence derive the four geodesic equations. In particular, show that

$$\ddot{v} + 2\left(\frac{\partial H}{\partial x}\,\dot{x} + \frac{\partial H}{\partial y}\,\dot{y}\right)\dot{u} + \frac{\partial H}{\partial u}\,\dot{u}^2 = 0.$$

By comparing these equations with the standard form of the geodesic equation, show that $\Gamma^2_{13} = \partial H/\partial x$ and derive the other Christoffel symbols.

The Ricci tensor, R_{ab}, is defined by

$$R_{ab} = \Gamma^d_{ab,d} - \Gamma^d_{ad,b} + \Gamma^d_{df}\Gamma^f_{ba} - \Gamma^d_{bf}\Gamma^f_{da}\,.$$

By considering the case $a = 1, b = 1$, show that the vacuum Einstein field equations imply

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0\,.$$

38B Fluid Dynamics II

The steady two-dimensional boundary-layer equations for flow primarily in the x-direction are

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dP}{dx}+\mu\frac{\partial^2 u}{\partial y^2}\,,$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,.$$

A thin, steady, two-dimensional jet emerges from a point at the origin and flows along the x-axis in a fluid at rest far from the x-axis. Show that the momentum flux

$$F=\int_{-\infty}^{\infty}\rho u^2\,dy$$

is independent of position x along the jet. Deduce that the thickness $\delta(x)$ of the jet increases along the jet as $x^{2/3}$, while the centre-line velocity $U(x)$ decreases as $x^{-1/3}$.

A similarity solution for the jet is sought with a streamfunction ψ of the form

$$\psi(x,y)=U(x)\delta(x)f(\eta)\quad\text{with}\quad \eta=y/\delta(x)\,.$$

Derive the nonlinear third-order non-dimensional differential equation governing f, and write down the boundary and normalisation conditions which must be applied.

39B Waves

An inviscid fluid with sound speed c_0 occupies the region $0<y<\pi\alpha$, $0<z<\pi\beta$ enclosed by the rigid boundaries of a rectangular waveguide. Starting with the acoustic wave equation, find the dispersion relation $\omega(k)$ for the propagation of sound waves in the x-direction.

Hence find the phase speed $c(k)$ and the group velocity $c_g(k)$ of both the dispersive modes and the nondispersive mode, and sketch the form of the results for $k,\omega>0$.

Define the time and cross-sectional average appropriate for a mode with frequency ω. For each dispersive mode, show that the average kinetic energy is equal to the average compressive energy.

A general multimode acoustic disturbance is created within the waveguide at $t=0$ in a region around $x=0$. Explain briefly how the amplitude of the disturbance varies with time as $t\to\infty$ at the moving position $x=Vt$ for each of the cases $0<V<c_0$, $V=c_0$ and $V>c_0$. [You may quote without proof any generic results from the method of stationary phase.]

40A Numerical Analysis

The nine-point method for the Poisson equation $\nabla^2 u = f$ (with zero Dirichlet boundary conditions) in a square, reads

$$\frac{2}{3}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right) + \frac{1}{6}\left(u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right) - \frac{10}{3}u_{i,j} = h^2 f_{i,j}, \quad i,j = 1,\ldots,m,$$

where $u_{0,j} = u_{m+1,j} = u_{i,0} = u_{i,m+1} = 0$, for all $i,j = 0,\ldots,m+1$.

(i) By arranging the two-dimensional arrays $\{u_{i,j}\}_{i,j=1,\ldots,m}$ and $\{f_{i,j}\}_{i,j=1,\ldots,m}$ into column vectors $u \in \mathbb{R}^{m^2}$ and $b \in \mathbb{R}^{m^2}$ respectively, the linear system above takes the matrix form $Au = b$. Prove that, regardless of the ordering of the points on the grid, the matrix A is symmetric and negative definite.

(ii) Formulate the Jacobi method with relaxation for solving the above linear system.

(iii) Prove that the iteration converges if the relaxation parameter ω is equal to 1.

[You may quote without proof any relevant result about convergence of iterative methods.]

END OF PAPER