UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 6 June, 2011 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

What does it mean to say that a function $f : X \to Y$ has an inverse? Show that a function has an inverse if and only if it is a bijection.

Let f and g be functions from a set X to itself. Which of the following are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) If f and g are bijections then $f \circ g$ is a bijection.

(ii) If $f \circ g$ is a bijection then f and g are bijections.

2E Numbers and Sets

What is an *equivalence relation* on a set X? If $\sim$ is an equivalence relation on X, what is an *equivalence class* of $\sim$? Prove that the equivalence classes of $\sim$ form a partition of X.

Let $\sim$ be the relation on the positive integers defined by $x \sim y$ if either x divides y or y divides x. Is $\sim$ an equivalence relation? Justify your answer.

Write down an equivalence relation on the positive integers that has exactly four equivalence classes, of which two are infinite and two are finite.

3B Dynamics and Relativity

The motion of a planet in the gravitational field of a star of mass M obeys

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{h^2}{r^3} = -\frac{GM}{r^2} , \qquad r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = h ,$$

where $r(t)$ and $\theta(t)$ are polar coordinates in a plane and h is a constant. Explain one of Kepler's Laws by giving a geometrical interpretation of h.

Show that circular orbits are possible, and derive another of Kepler's Laws relating the radius a and the period T of such an orbit. Show that any circular orbit is stable under small perturbations that leave h unchanged.

4B Dynamics and Relativity

Inertial frames S and S' in two-dimensional space-time have coordinates (x,t) and (x',t'), respectively. These coordinates are related by a Lorentz transformation with v the velocity of S' relative to S. Show that if $x_\pm = x \pm ct$ and $x'_\pm = x' \pm ct'$ then the Lorentz transformation can be expressed in the form

$$x'_+ = \lambda(v)x_+ \quad \text{and} \quad x'_- = \lambda(-v)x_- \,, \quad \text{where} \quad \lambda(v) = \left(\frac{c-v}{c+v}\right)^{1/2} . \qquad (*)$$

Deduce that $x^2 - c^2t^2 \;=\; x'^{\,2} - c^2t'^{\,2}$.

Use the form $(*)$ to verify that successive Lorentz transformations with velocities v_1 and v_2 result in another Lorentz transformation with velocity v_3, to be determined in terms of v_1 and v_2.

SECTION II

5E Numbers and Sets

(a) What is the *highest common factor* of two positive integers a and b? Show that the highest common factor may always be expressed in the form $\lambda a + \mu b$, where λ and μ are integers.

Which positive integers n have the property that, for any positive integers a and b, if n divides ab then n divides a or n divides b? Justify your answer.

Let a, b, c, d be distinct prime numbers. Explain carefully why ab cannot equal cd.

[*No form of the Fundamental Theorem of Arithmetic may be assumed without proof.*]

(b) Now let S be the set of positive integers that are congruent to 1 mod 10. We say that $x \in S$ is *irreducible* if $x > 1$ and whenever $a, b \in S$ satisfy $ab = x$ then $a = 1$ or $b = 1$. Do there exist distinct irreducibles a, b, c, d with $ab = cd$?

6E Numbers and Sets

State Fermat's Theorem and Wilson's Theorem.

Let p be a prime.

(a) Show that if $p \equiv 3 \pmod 4$ then the equation $x^2 \equiv -1 \pmod p$ has no solution.

(b) By considering $\left(\frac{p-1}{2}\right)!$, or otherwise, show that if $p \equiv 1 \pmod 4$ then the equation $x^2 \equiv -1 \pmod p$ does have a solution.

(c) Show that if $p \equiv 2 \pmod 3$ then the equation $x^3 \equiv -1 \pmod p$ has no solution other than $-1 \pmod p$.

(d) Using the fact that $14^2 \equiv -3 \pmod{199}$, find a solution of $x^3 \equiv -1 \pmod{199}$ that is not $-1 \pmod{199}$.

[*Hint: how are the complex numbers $\sqrt{-3}$ and $\sqrt[3]{-1}$ related?*]

7E Numbers and Sets

Define the binomial coefficient $\binom{n}{i}$, where n is a positive integer and i is an integer with $0 \leqslant i \leqslant n$. Arguing from your definition, show that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Prove the binomial theorem, that $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$ for any real number x.

By differentiating this expression, or otherwise, evaluate $\sum_{i=0}^{n} i \binom{n}{i}$ and $\sum_{i=0}^{n} i^2 \binom{n}{i}$.

By considering the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$, or otherwise, show that

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

Show that $\sum_{i=0}^{n} i \binom{n}{i}^2 = \frac{n}{2}\binom{2n}{n}$.

8E Numbers and Sets

Show that, for any set X, there is no surjection from X to the power-set of X.

Show that there exists an injection from $\mathbb{R}^2$ to $\mathbb{R}$.

Let A be a subset of $\mathbb{R}^2$. A *section* of A is a subset of $\mathbb{R}$ of the form

$$\{t \in \mathbb{R} : \ a + tb \in A\},$$

where $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^2$ with $b \neq 0$. Prove that there does not exist a set $A \subset \mathbb{R}^2$ such that every set $S \subset \mathbb{R}$ is a section of A.

Does there exist a set $A \subset \mathbb{R}^2$ such that every countable set $S \subset \mathbb{R}$ is a section of A? [*There is no requirement that every section of A should be countable.*] Justify your answer.

9B Dynamics and Relativity

A particle with mass m and position $\mathbf{r}(t)$ is subject to a force

$$\mathbf{F} = \mathbf{A}(\mathbf{r}) + \dot{\mathbf{r}}\times\mathbf{B}(\mathbf{r}) \,.$$

(a) Suppose that $\mathbf{A} = -\boldsymbol{\nabla}\phi$. Show that

$$E = \frac{1}{2}m\,\dot{\mathbf{r}}^2 + \phi(\mathbf{r})$$

is constant, and interpret this result, explaining why the field $\mathbf{B}$ plays no role.

(b) Suppose, in addition, that $\mathbf{B} = -\boldsymbol{\nabla}\psi$ and that both ϕ and ψ depend only on $r = |\mathbf{r}|$. Show that

$$\mathbf{L} = m\,\mathbf{r}\times\dot{\mathbf{r}} - \psi\mathbf{r}$$

is independent of time if $\psi(r) = \mu/r$, for any constant μ.

(c) Now specialise further to the case $\psi = 0$. Explain why the result in (b) implies that the motion of the particle is confined to a plane. Show also that

$$\mathbf{K} = \mathbf{L}\times\dot{\mathbf{r}} - \phi\mathbf{r}$$

is constant provided $\phi(r)$ takes a certain form, to be determined.

[*Recall that* $\mathbf{r}\cdot\dot{\mathbf{r}} = r\dot{r}$ *and that if* f *depends only on* $r = |\mathbf{r}|$ *then* $\boldsymbol{\nabla} f = f'(r)\hat{\mathbf{r}}$.]

10B Dynamics and Relativity

The trajectory of a particle $\mathbf{r}(t)$ is observed in a frame S which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame I. Given that the time derivative in I of any vector $\mathbf{u}$ is

$$\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right)_I = \dot{\mathbf{u}} + \boldsymbol{\omega}\times\mathbf{u} ,$$

where a dot denotes a time derivative in S, show that

$$m\ddot{\mathbf{r}} = \mathbf{F} - 2m\boldsymbol{\omega}\times\dot{\mathbf{r}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) ,$$

where $\mathbf{F}$ is the force on the particle and m is its mass.

Let S be the frame that rotates with the Earth. Assume that the Earth is a sphere of radius R. Let P be a point on its surface at latitude $\pi/2 - \theta$, and define vertical to be the direction normal to the Earth's surface at P.

(a) A particle at P is released from rest in S and is acted on only by gravity. Show that its initial acceleration makes an angle with the vertical of approximately

$$\frac{\omega^2 R}{g}\sin\theta\cos\theta ,$$

working to lowest non-trivial order in ω.

(b) Now consider a particle fired vertically upwards from P with speed v. Assuming that terms of order ω^2 and higher can be neglected, show that it falls back to Earth under gravity at a distance

$$\frac{4}{3}\frac{\omega v^3}{g^2}\sin\theta$$

from P. [*You may neglect the curvature of the Earth's surface and the vertical variation of gravity.*]

11B Dynamics and Relativity

A rocket carries equipment to collect samples from a stationary cloud of cosmic dust. The rocket moves in a straight line, burning fuel and ejecting gas at constant speed u relative to itself. Let $v(t)$ be the speed of the rocket, $M(t)$ its total mass, including fuel and any dust collected, and $m(t)$ the total mass of gas that has been ejected. Show that

$$M\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}M}{\mathrm{d}t} + (v-u)\frac{\mathrm{d}m}{\mathrm{d}t} = 0 \ ,$$

assuming that all external forces are negligible.

(a) If no dust is collected and the rocket starts from rest with mass M_0, deduce that

$$v = u \log(M_0/M) \ .$$

(b) If cosmic dust is collected at a constant rate of α units of mass per unit time and fuel is consumed at a constant rate $\mathrm{d}m/\mathrm{d}t = \beta$, show that, with the same initial conditions as in (a),

$$v = \frac{u\beta}{\alpha}\Big(1 - (M/M_0)^{\alpha/(\beta-\alpha)}\Big) \ .$$

Verify that the solution in (a) is recovered in the limit $\alpha \to 0$.

12B Dynamics and Relativity

(a) Write down the relativistic energy E of a particle of rest mass m and speed v. Find the approximate form for E when v is small compared to c, keeping all terms up to order $(v/c)^2$. What new physical idea (when compared to Newtonian Dynamics) is revealed in this approximation?

(b) A particle of rest mass m is fired at an identical particle which is at rest in the laboratory frame. Let E be the relativistic energy and v the speed of the incident particle in this frame. After the collision, there are N particles in total, each with rest mass m. Assuming that four-momentum is conserved, find a lower bound on E and hence show that

$$v \geqslant \frac{N(N^2-4)^{1/2}}{N^2-2}\, c \ .$$

END OF PAPER