UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday, 7 June, 2011 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

(a) Let G be the group of symmetries of the cube, and consider the action of G on the set of edges of the cube. Determine the stabilizer of an edge and its orbit. Hence compute the order of G.

(b) The symmetric group S_n acts on the set $X = \{1, \ldots, n\}$, and hence acts on $X \times X$ by $g(x, y) = (gx, gy)$. Determine the orbits of S_n on $X \times X$.

2D Groups

State and prove Lagrange's Theorem.

Show that the dihedral group of order $2n$ has a subgroup of order k for every k dividing $2n$.

3C Vector Calculus

Cartesian coordinates x, y, z and spherical polar coordinates r, θ, ϕ are related by

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta \ .$$

Find scalars h_r, h_θ, h_ϕ and unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ such that

$$\mathrm{d}\mathbf{x} = h_r \mathbf{e}_r \, \mathrm{d}r + h_\theta \mathbf{e}_\theta \, \mathrm{d}\theta + h_\phi \mathbf{e}_\phi \, \mathrm{d}\phi \ .$$

Verify that the unit vectors are mutually orthogonal.

Hence calculate the area of the open surface defined by $\theta = \alpha$, $0 \leqslant r \leqslant R$, $0 \leqslant \phi \leqslant 2\pi$, where α and R are constants.

4C Vector Calculus

State the value of $\partial x_i / \partial x_j$ and find $\partial r / \partial x_j$, where $r = |\mathbf{x}|$.

Vector fields $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ are given by $\mathbf{u} = r^\alpha \mathbf{x}$ and $\mathbf{v} = \mathbf{k} \times \mathbf{u}$, where α is a constant and $\mathbf{k}$ is a constant vector. Calculate the second-rank tensor $d_{ij} = \partial u_i / \partial x_j$, and deduce that $\boldsymbol{\nabla} \times \mathbf{u} = \mathbf{0}$ and $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$. When $\alpha = -3$, show that $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and

$$\boldsymbol{\nabla} \times \mathbf{v} = \frac{3(\mathbf{k} \cdot \mathbf{x})\mathbf{x} - \mathbf{k} r^2}{r^5} \ .$$

SECTION II

5D Groups

(a) Let G be a finite group, and let $g \in G$. Define the *order* of g and show it is finite. Show that if g is conjugate to h, then g and h have the same order.

(b) Show that every $g \in S_n$ can be written as a product of disjoint cycles. For $g \in S_n$, describe the order of g in terms of the cycle decomposition of g.

(c) Define the alternating group A_n. What is the condition on the cycle decomposition of $g \in S_n$ that characterises when $g \in A_n$?

(d) Show that, for every n, A_{n+2} has a subgroup isomorphic to S_n.

6D Groups

(a) Let

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \quad a, b, c, d \in \mathbb{Z} \right\},$$

and, for a prime p, let

$$SL_2(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \quad a, b, c, d \in \mathbb{F}_p \right\},$$

where $\mathbb{F}_p$ consists of the elements $0, 1, \ldots, p-1$, with addition and multiplication mod p.

Show that $SL_2(\mathbb{Z})$ and $SL_2(\mathbb{F}_p)$ are groups under matrix multiplication.

[*You may assume that matrix multiplication is associative, and that the determinant of a product equals the product of the determinants.*]

By defining a suitable homomorphism from $SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}_5)$, show that

$$\left\{ \begin{pmatrix} 1+5a & 5b \\ 5c & 1+5d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a, b, c, d \in \mathbb{Z} \right\}$$

is a normal subgroup of $SL_2(\mathbb{Z})$.

(b) Define the group $GL_2(\mathbb{F}_5)$, and show that it has order 480. By defining a suitable homomorphism from $GL_2(\mathbb{F}_5)$ to another group, which should be specified, show that the order of $SL_2(\mathbb{F}_5)$ is 120.

Find a subgroup of $GL_2(\mathbb{F}_5)$ of index 2.

7D Groups

(a) State the orbit–stabilizer theorem.

Let a group G act on itself by conjugation. Define the centre $Z(G)$ of G, and show that $Z(G)$ consists of the orbits of size 1. Show that $Z(G)$ is a normal subgroup of G.

(b) Now let $|G| = p^n$, where p is a prime and $n \geqslant 1$. Show that if G acts on a set X, and Y is an orbit of this action, then either $|Y| = 1$ or p divides $|Y|$.

Show that $|Z(G)| > 1$.

By considering the set of elements of G that commute with a fixed element x not in $Z(G)$, show that $Z(G)$ cannot have order p^{n-1}.

8D Groups

(a) Let G be a finite group and let H be a subgroup of G. Show that if $|G| = 2|H|$ then H is normal in G.

Show that the dihedral group D_{2n} of order $2n$ has a normal subgroup different from both D_{2n} and $\{e\}$.

For each integer $k \geqslant 3$, give an example of a finite group G, and a subgroup H, such that $|G| = k|H|$ and H is not normal in G.

(b) Show that A_5 is a simple group.

9C Vector Calculus

Write down the most general isotropic tensors of rank 2 and 3. Use the tensor transformation law to show that they are, indeed, isotropic.

Let V be the sphere $0 \leqslant r \leqslant a$. Explain briefly why

$$T_{i_1 \ldots i_n} = \int_V x_{i_1} \ldots x_{i_n} \, \mathrm{d}V$$

is an isotropic tensor for any n. Hence show that

$$\int_V x_i x_j \, \mathrm{d}V = \alpha \delta_{ij}, \quad \int_V x_i x_j x_k \, \mathrm{d}V = 0 \quad \text{and} \quad \int_V x_i x_j x_k x_l \, \mathrm{d}V = \beta(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

for some scalars α and β, which should be determined using suitable contractions of the indices or otherwise. Deduce the value of

$$\int_V \mathbf{x} \times (\boldsymbol{\Omega} \times \mathbf{x}) \, \mathrm{d}V \ ,$$

where $\boldsymbol{\Omega}$ is a constant vector.

[*You may assume that the most general isotropic tensor of rank 4 is*

$$\lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \nu \delta_{il}\delta_{jk} \ ,$$

where λ, μ and ν are scalars.]

10C Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{x})$ in a region V bounded by a piecewise smooth surface S with outward normal $\mathbf{n}$.

Show, by suitable choice of $\mathbf{u}$, that

$$\int_V \boldsymbol{\nabla} f \, \mathrm{d}V = \int_S f \, \mathrm{d}\mathbf{S} \qquad (*)$$

for a scalar field $f(\mathbf{x})$.

Let V be the paraboloidal region given by $z \geqslant 0$ and $x^2 + y^2 + cz \leqslant a^2$, where a and c are positive constants. Verify that $(*)$ holds for the scalar field $f = xz$.

11C Vector Calculus

The electric field $\mathbf{E}(\mathbf{x})$ due to a static charge distribution with density $\rho(\mathbf{x})$ satisfies

$$\mathbf{E} = -\boldsymbol{\nabla}\phi \,, \qquad \boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \,, \tag{1}$$

where $\phi(\mathbf{x})$ is the corresponding electrostatic potential and ε_0 is a constant.

(a) Show that the total charge Q contained within a closed surface S is given by Gauss' Law

$$Q = \varepsilon_0 \int_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} \,.$$

Assuming spherical symmetry, deduce the electric field and potential due to a point charge q at the origin i.e. for $\rho(\mathbf{x}) = q\,\delta(\mathbf{x})$.

(b) Let $\mathbf{E}_1$ and $\mathbf{E}_2$, with potentials ϕ_1 and ϕ_2 respectively, be the solutions to (1) arising from two different charge distributions with densities ρ_1 and ρ_2. Show that

$$\frac{1}{\varepsilon_0}\int_V \phi_1\rho_2 \,\mathrm{d}V + \int_{\partial V} \phi_1 \boldsymbol{\nabla}\phi_2 \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \phi_2\rho_1 \,\mathrm{d}V + \int_{\partial V} \phi_2 \boldsymbol{\nabla}\phi_1 \cdot \mathrm{d}\mathbf{S} \tag{2}$$

for any region V with boundary ∂V, where $\mathrm{d}\mathbf{S}$ points out of V.

(c) Suppose that $\rho_1(\mathbf{x}) = 0$ for $|\mathbf{x}| \leqslant a$ and that $\phi_1(\mathbf{x}) = \Phi$, a constant, on $|\mathbf{x}| = a$. Use the results of (a) and (b) to show that

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_{r>a} \frac{\rho_1(\mathbf{x})}{r}\,\mathrm{d}V \,.$$

[*You may assume that* $\phi_1 \to 0$ *as* $|\mathbf{x}| \to \infty$ *sufficiently rapidly that any integrals over the 'sphere at infinity' in* (2) *are zero.*]

12C Vector Calculus

The vector fields $\mathbf{A}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ obey the evolution equations

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \boldsymbol{\nabla}\psi \ , \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{B} \ , \tag{2}$$

where $\mathbf{u}$ is a given vector field and ψ is a given scalar field. Use suffix notation to show that the scalar field $h = \mathbf{A} \cdot \mathbf{B}$ obeys an evolution equation of the form

$$\frac{\partial h}{\partial t} = \mathbf{B} \cdot \boldsymbol{\nabla} f - \mathbf{u} \cdot \boldsymbol{\nabla} h \ ,$$

where the scalar field f should be identified.

Suppose that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. Show that, if $\mathbf{u} \cdot \mathbf{n} = \mathbf{B} \cdot \mathbf{n} = 0$ on the surface S of a fixed volume V with outward normal $\mathbf{n}$, then

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \ , \quad \text{where } H = \int_V h \, \mathrm{d}V \ .$$

Suppose that $\mathbf{A} = ar^2 \sin\theta \, \mathbf{e}_\theta + r(a^2 - r^2) \sin\theta \, \mathbf{e}_\phi$ with respect to spherical polar coordinates, and that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. Show that

$$h = ar^2(a^2 + r^2) \sin^2\theta \ ,$$

and calculate the value of H when V is the sphere $r \leqslant a$.

$$\left[\ \textit{In spherical polar coordinates } \boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta \mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} . \ \right]$$

END OF PAPER