15D Variational Principles

(i) Let $I[y] = \int_0^1 \big((y')^2 - y^2\big)dx$, where y is twice differentiable and $y(0) = y(1) = 0$. Write down the associated Euler-Lagrange equation and show that the only solution is $y(x) = 0$.

(ii) Let $J[y] = \int_0^1 (y' + y\tan x)^2 dx$, where y is twice differentiable and $y(0) = y(1) = 0$. Show that $J[y] = 0$ only if $y(x) = 0$.

(iii) Show that $I[y] = J[y]$ and deduce that the extremal value of $I[y]$ is a global minimum.

(iv) Use the second variation of $I[y]$ to verify that the extremal value of $I[y]$ is a local minimum.

(v) How would your answers to part (i) differ in the case $I[y] = \int_0^{2\pi} \big((y')^2 - y^2\big)dx$, where $y(0) = y(2\pi) = 0$? Show that the solution $y(x) = 0$ is not a global minimizer in this case. (You may use without proof the result $I[x(2\pi - x)] = -\frac{8\pi^3}{15}(2\pi^2 - 5)$.) Explain why the arguments of parts (iii) and (iv) cannot be used.