UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday, 4 June, 2010 9:00 am to 12:00 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A****,* ***B****,* ***C****, ...,* ***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Let p be a prime number, and put

$$a_k = kp\,, \quad N_k = a_k^p - 1 \quad (k = 1, 2, \ldots)\,.$$

Prove that a_k has exact order p modulo N_k for all $k \geqslant 1$, and deduce that N_k must be divisible by a prime q with $q \equiv 1 \pmod{p}$. By making a suitable choice of k, prove that there are infinitely many primes q with $q \equiv 1 \pmod{p}$.

2F Topics in Analysis

Find explicitly a polynomial p of degree $\leqslant 3$ such that

$$\sup_{x\in[-1,1]} |\,x^4 - p(x)| \;\leqslant\; \sup_{x\in[-1,1]} |\,x^4 - q(x)|$$

for every polynomial q of degree $\leqslant 3$. Justify your answer.

3F Geometry of Group Actions

Define *loxodromic transformations* and explain how to determine when a Möbius transformation

$$T : z \mapsto \frac{az+b}{cz+d} \qquad \text{with} \quad ad - bc = 1$$

is loxodromic.

Show that any Möbius transformation that maps a disc Δ onto itself cannot be loxodromic.

4H Coding and Cryptography

What is the discrete logarithm problem?

Describe the Diffie–Hellman key exchange system for two people. What is the connection with the discrete logarithm problem? Why might one use this scheme rather than just a public key system or a classical (pre-1960) coding system?

Extend the Diffie–Hellman system to n people using $n(n-1)$ transmitted numbers.

5J Statistical Modelling

Below is a simplified 1993 dataset of US cars. The columns list index, make, model, price (in \$1000), miles per gallon, number of passengers, length and width in inches, and weight (in pounds). The data are displayed in R as follows (abbreviated):

```
> cars
       make    model price mpg psngr length width weight
1     Acura  Integra  15.9  31     5    177    68   2705
2     Acura   Legend  33.9  25     5    195    71   3560
3      Audi       90  29.1  26     5    180    67   3375
4      Audi      100  37.7  26     6    193    70   3405
5       BMW     535i  30.0  30     4    186    69   3640
        ...              ...                         ...
92    Volvo      240  22.7  28     5    190    67   2985
93    Volvo      850  26.7  28     5    184    69   3245
```

It is reasonable to assume that prices for different makes of car are independent. We model the logarithm of the price as a linear combination of the other quantitative properties of the cars and an error term. Write down this model mathematically. How would you instruct R to fit this model and assign it to a variable “fit”?

R provides the following (slightly abbreviated) summary:

```
> summary(fit)

[...]

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  3.8751080  0.7687276   5.041 2.50e-06 ***
mpg         -0.0109953  0.0085475  -1.286 0.201724
psngr       -0.1782818  0.0290618  -6.135 2.45e-08 ***
length       0.0067382  0.0032890   2.049 0.043502 *
width       -0.0517544  0.0151009  -3.427 0.000933 ***
weight       0.0008373  0.0001302   6.431 6.60e-09 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
[...]
```

Briefly explain the information that is being provided in each column of the table. What are your conclusions and how would you try to improve the model?

6A Mathematical Biology

A concentration $u(x,t)$ obeys the differential equation

$$\frac{\partial u}{\partial t} = D u_{xx} + f(u) \ ,$$

in the domain $0 \leqslant x \leqslant L$, with boundary conditions $u(0,t) = u(L,t) = 0$ and initial condition $u(x,0) = u_0(x)$, and where D is a positive constant. Assume $f(0) = 0$ and $f'(0) > 0$. Linearising the dynamics around $u = 0$, and representing $u(x,t)$ as a suitable Fourier expansion, show that the condition for the linear stability of $u = 0$ can be expressed as the following condition on the domain length

$$L < \pi \left[\frac{D}{f'(0)} \right]^{1/2} .$$

7D Dynamical Systems

Consider the 2-dimensional flow

$$\dot{x} = y + \frac{1}{4} x \left(1 - 2x^2 - 2y^2 \right), \qquad \dot{y} = -x + \frac{1}{2} y \left(1 - x^2 - y^2 \right).$$

Use the Poincaré–Bendixson theorem, which should be stated carefully, to obtain a domain $\mathcal{D}$ in the xy-plane, within which there is at least one periodic orbit.

8E Further Complex Methods

The hypergeometric function $F(a,b;c;z)$ can be expressed in the form

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} (1-tz)^{-a} \, dt \,,$$

for appropriate restrictions on c, b, z.

Express the following integral in terms of a combination of hypergeometric functions

$$I(u,A) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{it(u+1)}}{e^{it} + iA} \, dt \,, \qquad |A| > 1 \,.$$

[You may use without proof that $\Gamma(z+1) = z\Gamma(z)$.]

9D Classical Dynamics

A system with one degree of freedom has Lagrangian $L(q,\dot{q})$. Define the canonical momentum p and the energy E. Show that E is constant along any classical path.

Consider a classical path $q_c(t)$ with the boundary–value data

$$q_c(0) = q_I\,, \qquad q_c(T) = q_F\,, \quad T > 0\,.$$

Define the action $S_c(q_I, q_F, T)$ of the path. Show that the total derivative dS_c/dT along the classical path obeys

$$\frac{dS_c}{dT} = L\,.$$

Using Lagrange's equations, or otherwise, deduce that

$$\frac{\partial S_c}{\partial q_F} = p_F\,, \qquad \frac{\partial S_c}{\partial T} = -E\,,$$

where p_F is the final momentum.

10D Cosmology

The linearised equation for the growth of density perturbations, $\delta_{\mathbf{k}}$, in an isotropic and homogenous universe is

$$\ddot{\delta}_{\mathbf{k}} + 2\,\frac{\dot{a}}{a}\,\dot{\delta}_{\mathbf{k}} + \left(\frac{{c_s}^2\,\mathbf{k}^2}{a^2} - 4\pi\,G\rho\right)\delta_{\mathbf{k}} = 0\,,$$

where ρ is the density of matter, c_s the sound speed, $c_s^2 = dP/d\rho$, and $\mathbf{k}$ is the comoving wavevector and $a(t)$ is the scale factor of the universe.

What is the Jean's length? Discuss its significance for the growth of perturbations.

Consider a universe filled with pressure-free matter with $a(t) = (t/t_0)^{2/3}$. Compute the resulting equation for the growth of density perturbations. Show that your equation has growing and decaying modes and comment briefly on the significance of this fact.

SECTION II

11G Number Theory

Let $\mathcal{S}$ be the set of all positive definite binary quadratic forms with integer coefficients. Define the action of the group $SL_2(\mathbb{Z})$ on $\mathcal{S}$, and prove that equivalent forms under this action have the same discriminant.

Find necessary and sufficient conditions for an odd positive integer n, prime to 35, to be properly represented by at least one of the two forms

$$x^2 + xy + 9y^2 , \qquad 3x^2 + xy + 3y^2 .$$

12F Geometry of Group Actions

Explain briefly how Möbius transformations of the Riemann sphere are extended to give isometries of the unit ball $B^3 \subset \mathbb{R}^3$ for the hyperbolic metric.

Which Möbius transformations have extensions that fix the origin in B^3?

For which Möbius transformations T can we find a hyperbolic line in B^3 that T maps onto itself? For which of these Möbius transformations is there only one such hyperbolic line?

13J Statistical Modelling

Every day, Barney the darts player comes to our laboratory. We record his facial expression, which can be either "mad", "weird" or "relaxed", as well as how many units of beer he has drunk that day. Each day he tries a hundred times to hit the bull's-eye, and we write down how often he succeeds. The data look like this:

```
>
Day Beer Expression  BullsEye
  1    3        Mad        30
  2    3        Mad        32
  ⋮    ⋮          ⋮         ⋮
 60    2        Mad        37
 61    4      Weird        30
  ⋮    ⋮          ⋮         ⋮
110    4      Weird        28
111    2    Relaxed        35
  ⋮    ⋮          ⋮         ⋮
150    3    Relaxed        31
```

Write down a reasonable model for $Y_1, \ldots, Y_n$, where $n = 150$ and where Y_i is the number of times Barney has hit bull's-eye on the ith day. Explain briefly why we may wish initially to include interactions between the variables. Write the R code to fit your model.

The scientist of the above story fitted her own generalized linear model, and subsequently obtained the following summary (abbreviated):

```
> summary(barney)
[...]

Coefficients:
                       Estimate Std. Error z value Pr(>|z|)
(Intercept)            -0.37258    0.05388  -6.916 4.66e-12 ***
Beer                   -0.09055    0.01595  -5.676 1.38e-08 ***
ExpressionWeird        -0.10005    0.08044  -1.244 0.213570
ExpressionRelaxed       0.29881    0.08268   3.614 0.000301 ***
Beer:ExpressionWeird    0.03666    0.02364   1.551 0.120933
Beer:ExpressionRelaxed -0.07697    0.02845  -2.705 0.006825 **

[...]
```

Why are `ExpressionMad` and `Beer:ExpressionMad` not listed? Suppose on a particular day, Barney's facial expression is weird, and he drank three units of beer. Give the linear predictor in the scientist's model for this day.

Based on the summary, how could you improve your model? How could one fit this new model in R (without modifying the data file)?

14D Dynamical Systems

Let $I = [0, 1]$ and consider continuous maps $F : I \to I$. Give an informal outline description of the two different bifurcations of fixed points of F that can occur.

Illustrate your discussion by considering in detail the logistic map

$$F(x) = \mu x (1 - x),$$

for $\mu \in (0, 1 + \sqrt{6}]$.

Describe qualitatively what happens for $\mu \in (1 + \sqrt{6}, 4]$.

[You may assume without proof that

$$x - F^2(x) = x (\mu x - \mu + 1) (\mu^2 x^2 - \mu(\mu + 1) x + \mu + 1).$$]

15D Classical Dynamics

A system is described by the Hamiltonian $H(q, p)$. Define the *Poisson bracket* $\{f, g\}$ of two functions $f(q, p, t)$, $g(q, p, t)$, and show from Hamilton's equations that

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

Consider the Hamiltonian

$$H = \frac{1}{2} (p^2 + \omega^2 q^2),$$

and define

$$a = (p - i\omega q)/(2\omega)^{1/2}, \qquad a^* = (p + i\omega q)/(2\omega)^{1/2},$$

where $i = \sqrt{-1}$. Evaluate $\{a, a\}$ and $\{a, a^*\}$, and show that $\{a, H\} = -i\omega a$ and $\{a^*, H\} = i\omega a^*$. Show further that, when $f(q, p, t)$ is regarded as a function of the independent complex variables a, a^* and of t, one has

$$\frac{df}{dt} = i\omega \left(a^* \frac{\partial f}{\partial a^*} - a \frac{\partial f}{\partial a} \right) + \frac{\partial f}{\partial t}.$$

Deduce that both $\log a^* - i\omega t$ and $\log a + i\omega t$ are constant during the motion.

16G Logic and Set Theory

State and prove the Completeness Theorem for Propositional Logic.

[You do **not** need to give definitions of the various terms involved. You may assume that the set of primitive propositions is countable. You may also assume the Deduction Theorem.]

Explain briefly how your proof should be modified if the set of primitive propositions is allowed to be uncountable.

17F Graph Theory

State Euler's formula relating the number of vertices, edges and faces in a drawing of a connected planar graph. Deduce that every planar graph has chromatic number at most 5.

Show also that any triangle-free planar graph has chromatic number at most 4.

Suppose G is a planar graph which is *minimal* 5-*chromatic;* that is to say, $\chi(G) = 5$ but if H is a subgraph of G with $H \neq G$ then $\chi(H) < 5$. Prove that $\delta(G) \geqslant 5$. Does this remain true if we drop the assumption that G is planar? Justify your answer.

[The Four Colour Theorem may not be assumed.]

18H Galois Theory

Let K be a field of characteristic $\neq 2, 3$, and assume that K contains a primitive cubic root of unity ζ. Let $P \in K[X]$ be an irreducible cubic polynomial, and let α, β, γ be its roots in the splitting field F of P over K. Recall that the Lagrange resolvent x of P is defined as $x = \alpha + \zeta\beta + \zeta^2\gamma$.

(i) List the possibilities for the group $\mathrm{Gal}(F/K)$, and write out the set $\{\sigma(x) \mid \sigma \in \mathrm{Gal}(F/K)\}$ in each case.

(ii) Let $y = \alpha + \zeta\gamma + \zeta^2\beta$. Explain why x^3, y^3 must be roots of a quadratic polynomial in $K[X]$. Compute this polynomial for $P = X^3 + bX + c$, and deduce the criterion to identify $\mathrm{Gal}(F/K)$ through the element $-4b^3 - 27c^2$ of K.

19F Representation Theory

Define the circle group $U(1)$. Give a complete list of the irreducible representations of $U(1)$.

Define the spin group $G = SU(2)$, and explain briefly why it is homeomorphic to the unit 3-sphere in $\mathbb{R}^4$. Identify the conjugacy classes of G and describe the classification of the irreducible representations of G. Identify the characters afforded by the irreducible representations. You need not give detailed proofs but you should define all the terms you use.

Let G act on the space $\mathrm{M}_3(\mathbb{C})$ of 3×3 complex matrices by conjugation, where $A \in SU(2)$ acts by

$$A : M \mapsto A_1 M A_1{}^{-1} ,$$

in which A_1 denotes the 3×3 block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Show that this gives a representation of G and decompose it into irreducibles.

20G Number Fields

Suppose that α is a zero of $x^3 - x + 3$ and that $K = \mathbb{Q}(\alpha)$. Show that $[K : \mathbb{Q}] = 3$. Show that O_K, the ring of integers in K, is $O_K = \mathbb{Z}[\alpha]$.

[You may quote any general theorem that you wish, provided that you state it clearly. Note that the discriminant of $x^3 + px + q$ is $-4p^3 - 27q^2$.]

21H Algebraic Topology

State the Snake Lemma. Explain how to define the boundary map which appears in it, and check that it is well-defined. Derive the Mayer-Vietoris sequence from the Snake Lemma.

Given a chain complex C, let $A \subset C$ be the span of all elements in C with grading greater than or equal to n, and let $B \subset C$ be the span of all elements in C with grading less than n. Give a short exact sequence of chain complexes relating A, B, and C. What is the boundary map in the corresponding long exact sequence?

22H Linear Analysis

Let X be a Banach space.

a) What does it mean for a bounded linear map $T : X \to X$ to be compact?

b) Let $\mathcal{B}(X)$ be the Banach space of all bounded linear maps $S : X \to X$. Let $\mathcal{B}_0(X)$ be the subset of $\mathcal{B}(X)$ consisting of all compact operators. Show that $\mathcal{B}_0(X)$ is a closed subspace of $\mathcal{B}(X)$. Show that, if $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}_0(X)$, then $ST, TS \in \mathcal{B}_0(X)$.

c) Let

$$X = \ell^2 = \left\{ x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \quad \text{and} \quad \|x\|_2^2 = \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\},$$

and $T : X \to X$ be defined by

$$(Tx)_k = \frac{x_{k+1}}{k+1}.$$

Is T compact? What is the spectrum of T? Explain your answers.

23G Algebraic Geometry

Let $E \subseteq \mathbf{P}^2$ be the projective curve obtained from the affine curve $y^2 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$, where the λ_i are distinct and $\lambda_1 \lambda_2 \lambda_3 \neq 0$.

(i) Show there is a unique point at infinity, P_∞.

(ii) Compute $\operatorname{div}(x)$, $\operatorname{div}(y)$.

(iii) Show $\mathcal{L}(P_\infty) = k$.

(iv) Compute $l(nP_\infty)$ for all n.

[You may *not* use the Riemann–Roch theorem.]

24H Differential Geometry

(i) Let $S \subset \mathbb{R}^3$ be a regular surface. Define the notions *exponential map*, *geodesic polar coordinates*, *geodesic circles*.

(ii) State and prove *Gauss' lemma*.

(iii) Let S be a regular surface. For fixed $r > 0$, and points p, q in S, let $S_r(p)$, $S_r(q)$ denote the geodesic circles around p, q, respectively, of radius r. Show the following statement: for each $p \in S$, there exists an $r = r(p) > 0$ and a neighborhood $\mathcal{U}_p$ containing p such that for all $q \in \mathcal{U}_p$, the sets $S_r(p)$ and $S_r(q)$ are smooth 1-dimensional manifolds which intersect transversally. What is the cardinality mod 2 of $S_r(p) \cap S_r(q)$?

25I Probability and Measure

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent normal random variables having mean 0 and variance 1. Set $S_n = X_1 + \ldots + X_n$ and $U_n = S_n - \lfloor S_n \rfloor$. Thus U_n is the fractional part of S_n. Show that U_n converges to U in distribution, as $n \to \infty$ where U is uniformly distributed on $[0, 1]$.

26I Applied Probability

(a) Let (X_t) be an irreducible continuous-time Markov chain on a finite or countable state space. What does it mean to say that the chain is (i) transient, (ii) recurrent, (iii) positive recurrent, (iv) null recurrent? What is the relation between equilibrium distributions and properties (iii) and (iv)?

A population of microorganisms develops in continuous time; the size of the population is a Markov chain (X_t) with states 0, 1, 2, ... Suppose $X_t = n$. It is known that after a short time s, the probability that X_t increased by one is $\lambda(n+1)s + o(s)$ and (if $n \geqslant 1$) the probability that the population was exterminated between times t and $t+s$ and never revived by time $t+s$ is $\mu s + o(s)$. Here λ and μ are given positive constants. All other changes in the value of X_t have a combined probability $o(s)$.

(b) Write down the Q-matrix of Markov chain (X_t) and determine if (X_t) is irreducible. Show that (X_t) is non-explosive. Determine the jump chain.

(c) Now assume that

$$\mu = \lambda .$$

Determine whether the chain is transient or recurrent, and in the latter case whether it is positive or null recurrent. Answer the same questions for the jump chain. Justify your answers.

27J Principles of Statistics

Define *completeness* and *bounded completeness* of a statistic T in a statistical experiment.

Random variables X_1, X_2, X_3 are generated as $X_i = \Theta^{1/2} Z + (1-\Theta)^{1/2} Y_i$, where Z, Y_1, Y_2, Y_3 are independently standard normal $\mathcal{N}(0,1)$, and the parameter Θ takes values in $(0,1)$. What is the joint distribution of (X_1, X_2, X_3) when $\Theta = \theta$? Write down its density function, and show that a minimal sufficient statistic for Θ based on (X_1, X_2, X_3) is $T = (T_1, T_2) := (\sum_{i=1}^3 X_i^2, (\sum_{i=1}^3 X_i)^2)$.

[*Hint: You may use that if I is the $n \times n$ identity matrix and J is the $n \times n$ matrix all of whose entries are 1, then $aI + bJ$ has determinant $a^{n-1}(a+nb)$, and inverse $cI + dJ$ with $c = 1/a$, $d = -b/(a(a+nb))$.*]

What is $\mathbb{E}_\theta(T_1)$? Is T complete for Θ?

Let $S := \mathrm{Prob}(X_1^2 \leqslant 1 \mid T)$. Show that $\mathbb{E}_\theta(S)$ is a positive constant c which does not depend on θ, but that S is not identically equal to c. Is T boundedly complete for Θ?

28J Optimization and Control

Dr Seuss' wealth x_t at time t evolves as

$$\frac{dx}{dt} = rx_t + \ell_t - c_t,$$

where $r > 0$ is the rate of interest earned, ℓ_t is his intensity of working ($0 \leqslant \ell \leqslant 1$), and c_t is his rate of consumption. His initial wealth $x_0 > 0$ is given, and his objective is to maximize

$$\int_0^T U(c_t, \ell_t)\, dt,$$

where $U(c, \ell) = c^\alpha (1-\ell)^\beta$, and T is the (fixed) time his contract expires. The constants α and β satisfy the inequalities $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta > 1$. At all times, c_t must be non-negative, and his final wealth x_T must be non-negative. Establish the following properties of the optimal solution (x^*, c^*, ℓ^*):

(i) $\beta c_t^* = \alpha(1 - \ell_t^*)$;

(ii) $c_t^* \propto e^{-\gamma r t}$, where $\gamma \equiv (\beta - 1 + \alpha)^{-1}$;

(iii) $x_t^* = Ae^{rt} + Be^{-\gamma r t} - r^{-1}$ for some constants A and B.

Hence deduce that the optimal wealth is

$$x_t^* = \frac{(1 - e^{-\gamma r T}(1 + rx_0))e^{rt} + ((1 + rx_0)e^{rT} - 1)e^{-\gamma r t}}{r(e^{rT} - e^{-\gamma r T})} - \frac{1}{r}.$$

29I Stochastic Financial Models

Consider a market with no riskless asset and d risky stocks where the price of stock $i \in \{1, \ldots, d\}$ at time $t \in \{0, 1\}$ is denoted S_t^i. We assume the vector $S_0 \in \mathbb{R}^d$ is not random, and we let $\mu = \mathbb{E}S_1$ and $V = \mathbb{E}\left[(S_1 - \mu)(S_1 - \mu)^T\right]$. Assume V is not singular.

Suppose an investor has initial wealth $X_0 = x$, which he invests in the d stocks so that his wealth at time 1 is $X_1 = \pi^T S_1$ for some $\pi \in \mathbb{R}^d$. He seeks to minimize the $\operatorname{var}(X_1)$ subject to his budget constraint and the condition that $\mathbb{E}X_1 = m$ for a given constant $m \in \mathbb{R}$.

Illustrate this investor's problem by drawing a diagram of the mean-variance efficient frontier. Write down the Lagrangian for the problem. Show that there are two vectors π_A and π_B (which do not depend on the constants x and m) such that the investor's optimal portfolio is a linear combination of π_A and π_B.

Another investor with initial wealth $Y_0 = y$ seeks to maximize $\mathbb{E}U(Y_1)$ his expected utility of time 1 wealth, subject to his budget constraint. Assuming that S_1 is Gaussian and $U(w) = -e^{-\gamma w}$ for a constant $\gamma > 0$, show that the optimal portfolio in this case is also a linear combination of π_A and π_B.

[You may use the moment generating function of the Gaussian distribution without derivation.]

Continue to assume S_1 is Gaussian, but now assume that U is increasing, concave, and twice differentiable, and that U, U' and U'' are of exponential growth but not necessarily of the form $U(w) = -e^{-\gamma w}$. (Recall that a function f is of exponential growth if $|f(w)| \leqslant a e^{b|w|}$ for some constants positive constants a, b.) Prove that the utility maximizing investor still holds a linear combination of π_A and π_B.

[You may use the Gaussian integration by parts formula

$$\mathbb{E}\left[\nabla f(Z)\right] = \mathbb{E}\left[Z f(Z)\right]$$

where $Z = (Z_1, \ldots, Z_d)^T$ is a vector of independent standard normal random variables, and f is differentiable and of exponential growth. You may also interchange integration and differentiation without justification.]

30E Partial Differential Equations

a) Solve the Dirichlet problem for the Laplace equation in a disc in $\mathbb{R}^2$

$$\begin{aligned} \Delta u = 0 \quad \text{in} \quad G = \{x^2 + y^2 < R^2\} \subseteq \mathbb{R}^2 \ , \ R > 0 \, , \\ u = u_D \quad \text{on} \quad \partial G \, , \end{aligned}$$

using polar coordinates (r, φ) and separation of variables, $u(x, y) = R(r)\Theta(\varphi)$. Then use the ansatz $R(r) = r^\alpha$ for the radial function.

b) Solve the Dirichlet problem for the Laplace equation in a square in $\mathbb{R}^2$

$$\begin{aligned} &\Delta u = 0 \quad \text{in} \quad G = [0, a] \times [0, a] \, , \\ &u(x, 0) = f_1(x) \, , \qquad u(x, a) = f_2(x) \, , \qquad u(0, y) = f_3(y) \, , \qquad u(a, y) = f_4(y) \, . \end{aligned}$$

31C Asymptotic Methods

(a) Consider for $\lambda > 0$ the Laplace type integral

$$I(\lambda) \; = \; \int_a^b f(t) \, \mathrm{e}^{-\lambda \phi(t)} \, dt \, ,$$

for some finite $a, b \in \mathbb{R}$ and smooth, real-valued functions $f(t), \phi(t)$. Assume that the function $\phi(t)$ has a single minimum at $t = c$ with $a < c < b$. Give an account of Laplace's method for finding the leading order asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$ and briefly discuss the difference if instead $c = a$ or $c = b$, i.e. when the minimum is attained at the boundary.

(b) Determine the leading order asymptotic behaviour of

$$I(\lambda) \; = \; \int_{-2}^{1} \cos t \, \mathrm{e}^{-\lambda t^2} \, dt \, , \qquad (*)$$

as $\lambda \to \infty$.

(c) Determine also the leading order asymptotic behaviour when $\cos t$ is replaced by $\sin t$ in $(*)$.

32C Principles of Quantum Mechanics

The Hamiltonian for a quantum system in the Schrödinger picture is

$$H_0 + V(t)\,,$$

where H_0 is independent of time. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Let $|a\rangle$ and $|b\rangle$ be orthonormal eigenstates of H_0 with eigenvalues E_a and E_b respectively. Assume $V(t) = 0$ for $t \leqslant 0$. Show that if the system is initially, at $t = 0$, in the state $|a\rangle$ then the probability of measuring it to be the state $|b\rangle$ after a time t is

$$\frac{1}{\hbar^2}\left|\int_0^t dt' \langle b|V(t')|a\rangle e^{\,i(E_b - E_a)t'/\hbar}\right|^2 \qquad (*)$$

to order $V(t)^2$.

Suppose a system has a basis of just two orthonormal states $|1\rangle$ and $|2\rangle$, with respect to which

$$H_0 \;=\; E\,I\,, \qquad V(t) = vt\,\sigma_1\,, \quad t \geqslant 0\,,$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Use $(*)$ to calculate the probability of a transition from state $|1\rangle$ to state $|2\rangle$ after a time t to order v^2.

Show that the time dependent Schrödinger equation has a solution

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\Big(Et\,I + \tfrac{1}{2}vt^2\,\sigma_1\Big)\right)|\psi(0)\rangle\,.$$

Calculate the transition probability exactly. Hence find the condition for the order v^2 approximation to be valid.

33B Applications of Quantum Mechanics

The scattering amplitude for electrons of momentum $\hbar\mathbf{k}$ incident on an atom located at the origin is $f(\hat{\mathbf{r}})$ where $\hat{\mathbf{r}} = \mathbf{r}/r$. Explain why, if the atom is displaced by a position vector $\mathbf{a}$, the asymptotic form of the scattering wave function becomes

$$\psi_{\mathbf{k}}(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}} + e^{i\mathbf{k}\cdot\mathbf{a}} \frac{e^{ikr'}}{r'} f(\hat{\mathbf{r}}') \sim e^{i\mathbf{k}\cdot\mathbf{r}} + e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}} \frac{e^{ikr}}{r} f(\hat{\mathbf{r}}) ,$$

where $\mathbf{r}' = \mathbf{r} - \mathbf{a}$, $r' = |\mathbf{r}'|$, $\hat{\mathbf{r}}' = \mathbf{r}'/r'$ and $k = |\mathbf{k}|$, $\mathbf{k}' = k\hat{\mathbf{r}}$. For electrons incident on N atoms in a regular Bravais crystal lattice show that the differential cross-section for scattering in the direction $\hat{\mathbf{r}}$ is

$$\frac{d\sigma}{d\Omega} = N\,|f(\hat{\mathbf{r}})|^2\,\Delta(\mathbf{k}-\mathbf{k}') .$$

Derive an explicit form for $\Delta(\mathbf{Q})$ and show that it is strongly peaked when $\mathbf{Q} \approx \mathbf{b}$ for $\mathbf{b}$ a reciprocal lattice vector.

State the Born approximation for $f(\hat{\mathbf{r}})$ when the scattering is due to a potential $V(\mathbf{r})$. Calculate the Born approximation for the case $V(\mathbf{r}) = -a\,\delta(\mathbf{r})$.

Electrons with de Broglie wavelength λ are incident on a target composed of many randomly oriented small crystals. They are found to be scattered strongly through an angle of 60°. What is the likely distance between planes of atoms in the crystal responsible for the scattering?

34C Statistical Physics

(i) Let ρ_i be the probability that a system is in a state labelled by i with N_i particles and energy E_i. Define

$$s(\rho_i) = -k \sum_i \rho_i \log \rho_i \,.$$

$s(\rho_i)$ has a maximum, consistent with a fixed mean total number of particles N, mean total energy E and $\sum_i \rho_i = 1$, when $\rho_i = \bar{\rho}_i$. Let $S(E,N) = s(\bar{\rho}_i)$ and show that

$$\frac{\partial S}{\partial E} = \frac{1}{T} \,, \qquad \frac{\partial S}{\partial N} = -\frac{\mu}{T} \,,$$

where T may be identified with the temperature and μ with the chemical potential.

(ii) For two weakly coupled systems 1,2 then $\rho_{i,j} = \rho_{1,i}\,\rho_{2,j}$ and $E_{i,j} = E_{1,i} + E_{2,j}$, $N_{i,j} = N_{1,i} + N_{2,j}$. Show that $S(E,N) = S_1(E_1,N_1) + S_2(E_2,N_2)$ where, if $S(E,N)$ is stationary under variations in E_1, E_2 and N_1, N_2 for $E = E_1 + E_2$, $N = N_1 + N_2$ fixed, we must have $T_1 = T_2$, $\mu_1 = \mu_2$.

(iii) Define the grand partition function $\mathcal{Z}(T,\mu)$ for the system in (i) and show that

$$k \log \mathcal{Z} = S - \frac{1}{T}\, E + \frac{\mu}{T}\, N \,, \qquad S = \frac{\partial}{\partial T}\big(kT \log \mathcal{Z}\big) \,.$$

(iv) For a system with single particle energy levels ϵ_r the possible states are labelled by $i = \{n_r : n_r = 0,1\}$, where $N_i = \sum_r n_r$, $E_i = \sum_r n_r \epsilon_r$ and $\sum_i = \prod_r \sum_{n_r=0,1}$. Show that

$$\bar{\rho}_i = \prod_r \frac{e^{-n_r(\epsilon_r - \mu)/kT}}{1 + e^{-(\epsilon_r - \mu)/kT}} \,.$$

Calculate $\bar{n}_r$. How is this related to a free fermion gas?

35B Electrodynamics

In a superconductor the number density of charge carriers of charge q is n_s . Suppose that there is a time-independent magnetic field described by the three-vector potential $\mathbf{A}$.

Derive an expression for the superconducting current.

Explain how your answer is gauge invariant.

Suppose that for $z < 0$ there is a constant magnetic field $\mathbf{B}_0$ in a vacuum and, for $z > 0$, there is a uniform superconductor. Derive the magnetic field for $z > 0$.

36B General Relativity

The Schwarzschild line element is given by

$$ds^2 = -F dt^2 + F^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right),$$

where $F = 1 - r_s/r$ and r_s is the Schwarzschild radius. Obtain the equation of geodesic motion of photons moving in the equatorial plane, $\theta = \pi/2$, in the form

$$\left(\frac{dr}{d\tau} \right)^2 = E^2 - \frac{h^2 F}{r^2},$$

where τ is proper time, and E and h are constants whose physical significance should be indicated briefly.

Defining $u = 1/r$ show that light rays are determined by

$$\left(\frac{du}{d\phi} \right)^2 = \left(\frac{1}{b} \right)^2 - u^2 + r_s u^3,$$

where $b = h/E$ and r_s may be taken to be small. Show that, to zeroth order in r_s, a light ray is a straight line passing at distance b from the origin. Show that, to first order in r_s, the light ray is deflected through an angle $2r_s/b$. Comment briefly on some observational evidence for the result.

37A Fluid Dynamics II

An axisymmetric incompressible Stokes flow has the Stokes stream function $\Psi(R,\theta)$ in spherical polar coordinates (R,θ,ϕ). Give expressions for the components u_R, u_θ of the flow field in terms of Ψ. Show that the equation satisfied by Ψ is

$$\mathcal{D}^2(\mathcal{D}^2\Psi) = 0\,, \quad \text{where} \quad \mathcal{D}^2 = \frac{\partial^2}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \qquad (*)$$

Fluid is contained between the two spheres $R = a, R = b$, with $b \gg a$. The fluid velocity vanishes on the outer sphere, while on the inner sphere $u_R = U\cos\theta, u_\theta = 0$. It is assumed that Stokes flow applies.

(i) Show that the Stokes stream function,

$$\Psi(R,\theta) = a^2 U \sin^2\theta\left(A\left(\frac{a}{R}\right) + B\left(\frac{R}{a}\right) + C\left(\frac{R}{a}\right)^2 + D\left(\frac{R}{a}\right)^4\right)\,,$$

is the general solution of $(*)$ proportional to $\sin^2\theta$ and write down the conditions on A, B, C, D that allow all the boundary conditions to be satisfied.

(ii) Now let $b \to \infty$, with $|\mathbf{u}| \to 0$ as $R \to \infty$. Show that $A = B = 1/4$ with $C = D = 0$.

(iii) Show that when b/a is very large but finite, then the coefficients have the approximate form

$$C \approx -\frac{3}{8}\frac{a}{b}\,, \quad D \approx \frac{1}{8}\frac{a^3}{b^3}\,, \quad A \approx \frac{1}{4} - \frac{3}{16}\frac{a}{b}\,, \quad B \approx \frac{1}{4} + \frac{9}{16}\frac{a}{b}\,.$$

38A Waves

Starting from the equations for one-dimensional unsteady flow of an inviscid compressible fluid, show that it is possible to find Riemann invariants $u \pm Q$ that are constant on characteristics $C_{\pm}$ given by

$$\frac{dx}{dt} = u \pm c\,,$$

where $u(x,t)$ is the velocity of the fluid and $c(x,t)$ is the local speed of sound. Show that $Q = 2(c - c_0)/(\gamma - 1)$ for the case of a perfect gas with adiabatic equation of state $p = p_0(\rho/\rho_0)^{\gamma}$, where p_0 , ρ_0 and γ are constants, $\gamma > 1$ and $c = c_0$ when $\rho = \rho_0$.

Such a gas initially occupies the region $x > 0$ to the right of a piston in an infinitely long tube. The gas is initially uniform and at rest with density ρ_0 . At $t = 0$ the piston starts moving to the left at a constant speed V. Assuming that the gas keeps up with the piston, find $u(x,t)$ and $c(x,t)$ in each of the three distinct regions that are defined by families of C_+ characteristics.

Now assume that the gas does not keep up with the piston. Show that the gas particle at $x = x_0$ when $t = 0$ follows a trajectory given, for $t > x_0/c_0$, by

$$x(t) = \frac{\gamma+1}{\gamma-1}\left(\frac{c_0 t}{x_0}\right)^{2/(\gamma+1)} x_0 - \frac{2\,c_0 t}{\gamma-1}\,.$$

Deduce that the velocity of any given particle tends to $-2\,c_0/(\gamma - 1)$ as $t \to \infty$.

39A Numerical Analysis

An s-stage explicit Runge–Kutta method of order p, with constant step size $h > 0$, is applied to the differential equation $y' = \lambda\, y,\ t \geqslant 0$.

(a) Prove that

$$y_{n+1} = P_s(\lambda h)\, y_n\,.$$

where P_s is a polynomial of degree s.

(b) Prove that the order p of any s-stage explicit Runge–Kutta method satisfies the inequality $p \leqslant s$ and, for $p = s$, write down an explicit expression for P_s.

(c) Prove that no explicit Runge–Kutta method can be A-stable.

END OF PAPER