UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 3 June, 2010 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *. . .,* ***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

(i) Let M and N be positive integers, such that N is not a perfect square. If $M < \sqrt{N}$, show that every solution of the equation

$$x^2 - Ny^2 = M$$

in positive integers x, y comes from some convergent of the continued fraction of $\sqrt{N}$.

(ii) Find a solution in positive integers x, y of

$$x^2 - 29y^2 = 5\,.$$

2F Topics in Analysis

Let $A = \{z \in \mathbb{C} \,:\, 1/2 \leqslant |z| \leqslant 2\}$ and suppose that f is complex analytic on an open subset containing A.

(i) Give an example, with justification, to show that there need not exist a sequence of complex polynomials converging to f uniformly on A.

(ii) Let $R \subset \mathbb{C}$ be the positive real axis and $B = A \setminus R$. Prove that there exists a sequence of complex polynomials $p_1, p_2, p_3, \ldots$ such that $p_j \to f$ uniformly on each compact subset of B.

(iii) Let $p_1, p_2, p_3, \ldots$ be the sequence of polynomials in (ii). If this sequence converges uniformly on A, show that $\int_C f(z)\,dz = 0$, where $C = \{z \in \mathbb{C} \,:\, |z| = 1\}$.

3F Geometry of Group Actions

Let U be a "triangular" region in the unit disc $\mathbb{D}$ bounded by three hyperbolic geodesics $\gamma_1, \gamma_2, \gamma_3$ that do not meet in $\mathbb{D}$ nor on its boundary. Let J_k be inversion in γ_k and set

$$A = J_2 \circ J_1\,; \qquad B = J_3 \circ J_2\,.$$

Let G be the group generated by the Möbius transformations A and B. Describe briefly a fundamental set for the group G acting on $\mathbb{D}$.

Prove that G is a free group on the two generators A and B. Describe the quotient surface $\mathbb{D}/G$.

4H Coding and Cryptography

What is a linear code? What is a parity check matrix for a linear code? What is the minimum distance $d(C)$ for a linear code C?

If C_1 and C_2 are linear codes having a certain relation (which you should specify), define the bar product $C_1|\,C_2$. Show that

$$d(C_1|\,C_2) \;=\; \min\left\{2d(C_1), d(C_2)\right\}.$$

If C_1 has parity check matrix P_1 and C_2 has parity check matrix P_2, find a parity check matrix for $C_1|\,C_2$.

5J Statistical Modelling

Consider the linear model $Y = X\beta + \varepsilon$, where Y is a $n \times 1$ random vector, $\varepsilon \sim N_n(0, \sigma^2 I)$, and where the $n \times p$ nonrandom matrix X is known and has full column rank p. Derive the maximum likelihood estimator $\hat{\sigma}^2$ of σ^2. Without using Cochran's theorem, show carefully that $\hat{\sigma}^2$ is biased. Suggest another estimator $\tilde{\sigma}^2$ for σ^2 that is unbiased.

6A Mathematical Biology

A population of aerobic bacteria swims in a laterally-infinite layer of fluid occupying $-\infty < x < \infty$, $-\infty < y < \infty$, and $-d/2 < z < d/2$, with the top and bottom surfaces in contact with air. Assuming that there is no fluid motion and that all physical quantities depend only on z, the oxygen concentration c and bacterial concentration n obey the coupled equations

$$\frac{\partial c}{\partial t} = D_c \frac{\partial^2 c}{\partial z^2} - kn\,,$$
$$\frac{\partial n}{\partial t} = D_n \frac{\partial^2 n}{\partial z^2} - \frac{\partial}{\partial z}\left(\mu n \frac{\partial c}{\partial z}\right).$$

Consider first the case in which there is no chemotaxis, so n has the spatially-uniform value $\bar{n}$. Find the steady-state oxygen concentration consistent with the boundary conditions $c\,(\pm d/2) = c_0$. Calculate the Fick's law flux of oxygen into the layer and justify your answer on physical grounds.

Now allowing chemotaxis and cellular diffusion, show that the equilibrium oxygen concentration satisfies

$$\frac{d^2 c}{dz^2} - \frac{k n_0}{D_c} \exp\left(\mu c / D_n\right) \;=\; 0\,,$$

where n_0 is a suitable normalisation constant that need not be found.

7D Dynamical Systems

Let $I = [0,1)$. The sawtooth (Bernoulli shift) map $F : I \to I$ is defined by

$$F(x) = 2x \;\; [\bmod 1] .$$

Describe the effect of F using binary notation. Show that F is continuous on I except at $x = \frac{1}{2}$. Show also that F has N-periodic points for all $N \geqslant 2$. Are they stable?

Explain why F is chaotic, using Glendinning's definition.

8E Further Complex Methods

Let $\Gamma(z)$ and $\zeta(z)$ denote the gamma and the zeta functions respectively, namely

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx , \qquad \operatorname{Re} z > 0 ,$$

$$\zeta(z) = \sum_{m=1}^{\infty} \frac{1}{m^z} , \qquad \operatorname{Re} z > 1 .$$

By employing a series expansion of $(1 - e^{-x})^{-2}$, prove the following identity

$$\int_0^\infty \frac{x^z}{(e^x - 1)^2} \, dx = \Gamma(z+1) \big[\zeta(z) - \zeta(z+1) \big] , \qquad \operatorname{Re} z > 1 .$$

9D Classical Dynamics

Euler's equations for the angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ of a rigid body, viewed in the body frame, are

$$I_1 \frac{d\omega_1}{dt} = (I_2 - I_3)\,\omega_2\,\omega_3$$

and cyclic permutations, where the principal moments of inertia are assumed to obey $I_1 < I_2 < I_3$.

Write down two quadratic first integrals of the motion.

There is a family of solutions $\boldsymbol{\omega}(t)$, unique up to time–translations $t \to (t - t_0)$, which obey the boundary conditions $\boldsymbol{\omega} \to (0, \Omega, 0)$ as $t \to -\infty$ and $\boldsymbol{\omega} \to (0, -\Omega, 0)$ as $t \to \infty$, for a given positive constant Ω. Show that, for such a solution, one has

$$\mathbf{L}^2 = 2EI_2\,,$$

where $\mathbf{L}$ is the angular momentum and E is the kinetic energy.

By eliminating ω_1 and ω_3 in favour of ω_2, or otherwise, show that, in this case, the second Euler equation reduces to

$$\frac{ds}{d\tau} = 1 - s^2\,,$$

where $s = \omega_2/\Omega$ and $\tau = \Omega t\left[(I_1 - I_2)(I_2 - I_3)/I_1 I_3\right]^{1/2}$. Find the general solution $s(\tau)$.

[You are not expected to calculate $\omega_1(t)$ or $\omega_3(t)$.]

10D Cosmology

Consider a homogenous and isotropic universe with mass density $\rho(t)$, pressure $P(t)$ and scale factor $a(t)$. As the universe expands its energy changes according to the relation $dE = -PdV$. Use this to derive the fluid equation

$$\dot{\rho} = -3\,\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right).$$

Use conservation of energy applied to a test particle at the boundary of a spherical fluid element to derive the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi}{3}\,G\rho - \frac{k}{a^2}\,c^2\,,$$

where k is a constant. State any assumption you have made. Briefly state the significance of k.

SECTION II

11G Number Theory

State precisely the Miller-Rabin primality test.

(i) Let p be a prime $\geqslant 5$, and define

$$N = \frac{4^p - 1}{3}.$$

Prove that N is a composite odd integer, and that N is a pseudo-prime to the base 2.

(ii) Let M be an odd integer greater than 1 such that M is a pseudo-prime to the base 2. Prove that $2^M - 1$ is always a strong pseudo-prime to the base 2.

12F Topics in Analysis

(i) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map with $\gamma(0) = \gamma(1)$. Define the *winding number* $w(\gamma; 0)$ of γ about the origin.

(ii) For $j = 0, 1$, let $\gamma_j : [0,1] \to \mathbb{C} \setminus \{0\}$ be continuous with $\gamma_j(0) = \gamma_j(1)$. Make the following statement precise, and prove it: *if γ_0 can be continuously deformed into γ_1 through a family of continuous curves missing the origin, then $w(\gamma_0; 0) = w(\gamma_1; 0)$.*

[You may use without proof the following fact: if $\gamma, \delta : [0,1] \to \mathbb{C} \setminus \{0\}$ are continuous with $\gamma(0) = \gamma(1)$, $\delta(0) = \delta(1)$ and if $|\gamma(t)| < |\delta(t)|$ for each $t \in [0,1]$, then $w(\gamma + \delta; 0) = w(\delta; 0)$.]

(iii) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be continuous with $\gamma(0) = \gamma(1)$. If $\gamma(t)$ is not equal to a negative real number for each $t \in [0,1]$, prove that $w(\gamma; 0) = 0$.

(iv) Let $D = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and $C = \{z \in \mathbb{C} : |z| = 1\}$. If $g : D \to C$ is continuous, prove that for each non-zero integer n, there is at least one point $z \in C$ such that $z^n + g(z) = 0$.

13A Mathematical Biology

Consider an epidemic model in which $S(x,t)$ is the local population density of susceptibles and $I(x,t)$ is the density of infectives

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\,rIS\,,\\ \frac{\partial I}{\partial t} &= D\,\frac{\partial^2 I}{\partial x^2} + rIS - aI\,,\end{aligned}$$

where r, a, and D are positive. If S_0 is a characteristic population value, show that the rescalings $I/S_0 \to I$, $S/S_0 \to S$, $(rS_0/D)^{1/2}\,x \to x$, $rS_0t \to t$ reduce this system to

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\,IS\,,\\ \frac{\partial I}{\partial t} &= \frac{\partial^2 I}{\partial x^2} + IS - \lambda I\,,\end{aligned}$$

where λ should be found.

Travelling wavefront solutions are of the form $S(x,t) = S(z)$, $I(x,t) = I(z)$, where $z = x - ct$ and c is the wave speed, and we seek solutions with boundary conditions $S(\infty) = 1$, $S'(\infty) = 0$, $I(\infty) = I(-\infty) = 0$. Under the travelling-wave assumption reduce the rescaled PDEs to ODEs, and show by linearisation around the leading edge of the advancing front that the requirement that I be non-negative leads to the condition $\lambda < 1$ and hence the wave speed relation

$$c \geqslant 2(1-\lambda)^{1/2}\,, \qquad \lambda < 1\,.$$

Using the two ODEs you have obtained, show that the surviving susceptible population fraction $\sigma = S(-\infty)$ after the passage of the front satisfies

$$\sigma - \lambda \ln \sigma = 1\,,$$

and sketch σ as a function of λ.

14D Dynamical Systems

Describe informally the concepts of extended stable manifold theory. Illustrate your discussion by considering the 2-dimensional flow

$$\dot{x} = \mu x + xy - x^3\,, \qquad \dot{y} = -y + y^2 - x^2\,,$$

where μ is a parameter with $|\mu| \ll 1$, in a neighbourhood of the origin. Determine the nature of the bifurcation.

15D Cosmology

The number density for particles in thermal equilibrium, neglecting quantum effects, is

$$n = g_s \frac{4\pi}{h^3} \int p^2 \, dp \ \exp(-(E(p) - \mu)/kT) \,,$$

where g_s is the number of degrees of freedom for the particle with energy $E(p)$ and μ is its chemical potential. Evaluate n for a non-relativistic particle.

Thermal equilibrium between two species of non-relativistic particles is maintained by the reaction

$$a + \alpha \leftrightarrow b + \beta \,,$$

where α and β are massless particles. Evaluate the ratio of number densities n_a/n_b given that their respective masses are m_a and m_b and chemical potentials are μ_a and μ_b.

Explain how a reaction like the one above is relevant to the determination of the neutron to proton ratio in the early universe. Why does this ratio not fall rapidly to zero as the universe cools?

Explain briefly the process of primordial nucleosynthesis by which neutrons are converted into stable helium nuclei. Letting

$$Y_{He} = \rho_{He}/\rho$$

be the fraction of the universe's helium, compute Y_{He} as a function of the ratio $r = n_n/n_p$ at the time of nucleosynthesis.

16G Logic and Set Theory

Define the sets V_α , $\alpha \in ON$. What is meant by the *rank* of a set?

Explain briefly why, for every α , there exists a set of rank α .

Let x be a transitive set of rank α . Show that x has an element of rank β for every $\beta < \alpha$.

For which α does there exist a finite set of rank α? For which α does there exist a finite transitive set of rank α? Justify your answers.

[Standard properties of rank may be assumed.]

17F Graph Theory

Let G be a graph of order n. Show that G must contain an independent set of $\left\lceil \sum_{v \in G} \frac{1}{d(v)+1} \right\rceil$ vertices (where $\lceil x \rceil$ denotes the least integer $\geqslant x$).

[*Hint: take a random ordering of the vertices of G, and consider the set of those vertices which are adjacent to no earlier vertex in the ordering.*]

Fix an integer $m < n$ with m dividing n, and suppose that $e(G) = m\binom{n/m}{2}$.

(i) Deduce that G must contain an independent set of m vertices.

(ii) Must G contain an independent set of $m+1$ vertices?

18H Galois Theory

Let K be a field of characteristic 0. It is known that soluble extensions of K are contained in a succession of cyclotomic and Kummer extensions. We will refine this statement.

Let n be a positive integer. The n-th cyclotomic field over a field K is denoted by $K(\boldsymbol{\mu}_n)$. Let ζ_n be a primitive n-th root of unity in $K(\boldsymbol{\mu}_n)$.

(i) Write $\zeta_3 \in \mathbb{Q}(\boldsymbol{\mu}_3)$, $\zeta_5 \in \mathbb{Q}(\boldsymbol{\mu}_5)$ in terms of radicals. Write $\mathbb{Q}(\boldsymbol{\mu}_3)/\mathbb{Q}$ and $\mathbb{Q}(\boldsymbol{\mu}_5)/\mathbb{Q}$ as a succession of Kummer extensions.

(ii) Let $n > 1$, and $F := K(\zeta_1, \zeta_2, \ldots, \zeta_{n-1})$. Show that $F(\boldsymbol{\mu}_n)/F$ can be written as a succession of Kummer extensions, using the structure theorem of finite abelian groups (in other words, roots of unity can be written in terms of radicals). Show that every soluble extension of K is contained in a succession of Kummer extensions.

19F Representation Theory

Show that the degree of a complex irreducible character of a finite group is a factor of the order of the group.

State and prove Burnside's $p^a q^b$ theorem. You should quote clearly any results you use.

Prove that for any group of odd order n having precisely k conjugacy classes, the integer $n - k$ is divisible by 16.

20H Algebraic Topology

Suppose X is a finite simplicial complex and that $H_*(X)$ is a free abelian group for each value of $*$. Using the Mayer-Vietoris sequence or otherwise, compute $H_*(S^1 \times X)$ in terms of $H_*(X)$. Use your result to compute $H_*(T^n)$.

[Note that $T^n = S^1 \times \ldots \times S^1$, where there are n factors in the product.]

21H Linear Analysis

State and prove the Stone-Weierstrass theorem for real-valued functions.

[You may use without proof the fact that the function $s \to |s|$ can be uniformly approximated by polynomials on $[-1,1]$.]

22G Riemann Surfaces

Show that the analytic isomorphisms (i.e. conformal equivalences) of the Riemann sphere $\mathbb{C}_\infty$ to itself are given by the non-constant Möbius transformations.

State the Riemann–Hurwitz formula for a non-constant analytic map between compact Riemann surfaces, carefully explaining the terms which occur.

Suppose now that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an analytic map of degree 2; show that there exist Möbius transformations S and T such that

$$SfT : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

is the map given by $z \mapsto z^2$.

23G Algebraic Geometry

(i) Let X be a curve, and $p \in X$ be a smooth point on X. Define what a *local parameter* at p is.

Now let $f : X \dashrightarrow Y$ be a rational map to a quasi-projective variety Y. Show that if Y is projective, f extends to a morphism defined at p.

Give an example where this fails if Y is not projective, and an example of a morphism $f : \mathbb{C}^2 \setminus \{0\} \to \mathbf{P}^1$ which does not extend to 0.

(ii) Let $V = Z(X_0^8 + X_1^8 + X_2^8)$ and $W = Z(X_0^4 + X_1^4 + X_2^4)$ be curves in $\mathbf{P}^2$ over a field of characteristic not equal to 2. Let $\phi : V \to W$ be the map $[X_0 : X_1 : X_2] \mapsto [X_0^2 : X_1^2 : X_2^2]$. Determine the degree of ϕ, and the ramification e_p for all $p \in V$.

24H Differential Geometry

(i) State and prove the *Theorema Egregium.*

(ii) Define the notions *principal curvatures*, *principal directions* and *umbilical point.*

(iii) Let $S \subset \mathbb{R}^3$ be a connected compact regular surface (without boundary), and let $D \subset S$ be a dense subset of S with the following property. For all $p \in D$, there exists an open neighbourhood $\mathcal{U}_p$ of p in S such that for all $\theta \in [0, 2\pi)$, $\psi_{p,\theta}(\mathcal{U}_p) = \mathcal{U}_p$, where $\psi_{p,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes rotation by θ around the line through p perpendicular to $T_p S$. Show that S is in fact a sphere.

25I Probability and Measure

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables with common density function

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

Fix $\alpha \in [0,1]$ and set

$$Y_n = \operatorname{sgn}(X_n)|X_n|^\alpha, \quad S_n = Y_1 + \ldots + Y_n.$$

Show that for all $\alpha \in [0,1]$ the sequence of random variables S_n/n converges in distribution and determine the limit.

[*Hint: In the case $\alpha = 1$ it may be useful to prove that $\mathbb{E}(e^{iuX_1}) = e^{-|u|}$, for all $u \in \mathbb{R}$.*]

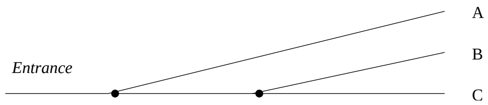
Show further that for all $\alpha \in [0, 1/2)$ the sequence of random variables $S_n/\sqrt{n}$ converges in distribution and determine the limit.

[You should state clearly any result about random variables from the course to which you appeal. You are not expected to evaluate explicitly the integral

$$m(\alpha) = \int_0^\infty \frac{x^\alpha}{\pi(1+x^2)} dx. \quad]$$

26I Applied Probability

Cars looking for a parking space are directed to one of three unlimited parking lots A, B and C. First, immediately after the entrance, the road forks: one direction is to lot A, the other to B and C. Shortly afterwards, the latter forks again, between B and C. See the diagram below.



The policeman at the first road fork directs an entering car with probability 1/3 to A and with probability 2/3 to the second fork. The policeman at the second fork sends the passing cars to B or C alternately: cars 1, 3, 5, ... approaching the second fork go to B and cars 2, 4, 6, ... to C.

Assuming that the total arrival process $(N(t))$ of cars is Poisson of rate λ, consider the processes $(X^{\mathrm{A}}(t))$, $(X^{\mathrm{B}}(t))$ and $(X^{\mathrm{C}}(t))$, $t \geqslant 0$, where $X^i(t)$ is the number of cars directed to lot i by time t, for $i =$A, B, C. The times for a car to travel from the first to the second fork, or from a fork to the parking lot, are all negligible.

(a) Characterise each of the processes $(X^{\mathrm{A}}(t))$, $(X^{\mathrm{B}}(t))$ and $(X^{\mathrm{C}}(t))$, by specifying if it is (i) Poisson, (ii) renewal or (iii) delayed renewal. Correspondingly, specify the rate, the holding-time distribution and the distribution of the delay.

(b) In the case of a renewal process, determine the equilibrium delay distribution.

(c) Given $s, t > 0$, write down explicit expressions for the probability $\mathbb{P}\big(X^i(s) = X^i(s+t)\big)$ that the interval $(s, t+s)$ is free of points in the corresponding process, $i = \mathrm{A}, \mathrm{B}, \mathrm{C}$.

27J Principles of Statistics

Define the *normal* and *extensive* form solutions of a Bayesian statistical decision problem involving parameter Θ, random variable X, and loss function $L(\theta, a)$. How are they related? Let $R_0 = R_0(\Pi)$ be the Bayes loss of the optimal act when $\Theta \sim \Pi$ and no data can be observed. Express the Bayes risk R_1 of the optimal statistical decision rule in terms of R_0 and the joint distribution of (Θ, X).

The real parameter Θ has distribution Π, having probability density function $\pi(\cdot)$. Consider the problem of specifying a set $S \subseteq \mathbb{R}$ such that the loss when $\Theta = \theta$ is $L(\theta, S) = c\,|S| - \mathbf{1}_S(\theta)$, where $\mathbf{1}_S$ is the indicator function of S, where $c > 0$, and where $|S| = \int_S dx$. Show that the "highest density" region $S^* := \{\theta : \pi(\theta) \geqslant c\}$ supplies a Bayes act for this decision problem, and explain why $R_0(\Pi) \leqslant 0$.

For the case $\Theta \sim \mathcal{N}(\mu, \sigma^2)$, find an expression for R_0 in terms of the standard normal distribution function Φ.

Suppose now that $c = 0.5$, that $\Theta \sim \mathcal{N}(0, 1)$ and that $X|\Theta \sim \mathcal{N}(\Theta, 1/9)$. Show that $R_1 < R_0$.

28J Optimization and Control

Consider an infinite-horizon controlled Markov process having per-period costs $c(x,u) \geqslant 0$, where $x \in \mathcal{X}$ is the state of the system, and $u \in \mathcal{U}$ is the control. Costs are discounted at rate $\beta \in (0,1]$, so that the objective to be minimized is

$$\mathbb{E}\Big[\sum_{t \geqslant 0} \beta^t c(X_t, u_t) \mid X_0 = x \Big].$$

What is meant by a *policy* π for this problem?

Let $\mathcal{L}$ denote the dynamic programming operator

$$\mathcal{L}f(x) \equiv \inf_{u \in \mathcal{U}} \big\{ \, c(x,u) + \beta \mathbb{E}\big[\, f(X_1) \, \big| \, X_0 = x, u_0 = u \, \big] \, \big\}.$$

Further, let F denote the value of the optimal control problem:

$$F(x) = \inf_{\pi} \mathbb{E}^{\pi}\Big[\sum_{t \geqslant 0} \beta^t c(X_t, u_t) \mid X_0 = x \Big],$$

where the infimum is taken over all policies π, and $\mathbb{E}^{\pi}$ denotes expectation under policy π. Show that the functions F_t defined by

$$F_{t+1} = \mathcal{L}F_t \quad (t \geqslant 0), \qquad F_0 \equiv 0$$

increase to a limit $F_\infty \in [0, \infty]$. Prove that $F_\infty \leqslant F$. Prove that $F = \mathcal{L}F$.

Suppose that $\Phi = \mathcal{L}\Phi \geqslant 0$. Prove that $\Phi \geqslant F$.

[You may assume that there is a function $u_* : \mathcal{X} \to \mathcal{U}$ such that

$$\mathcal{L}\Phi(x) = c(x, u_*(x)) + \beta \mathbb{E}\big[\, \Phi(X_1) \, \big| \, X_0 = x, u_0 = u_*(x) \, \big],$$

though the result remains true without this simplifying assumption.]

29I Stochastic Financial Models

Consider a market with two assets, a riskless bond and a risky stock, both of whose initial (time-0) prices are $B_0 = 1 = S_0$. At time 1, the price of the bond is a constant $B_1 = R > 0$ and the price of the stock S_1 is uniformly distributed on the interval $[0, C]$ where $C > R$ is a constant.

Describe the set of state price densities.

Consider a contingent claim whose payout at time 1 is given by S_1^2. Use the fundamental theorem of asset pricing to show that, if there is no arbitrage, the initial price of the claim is larger than R and smaller than C.

Now consider an investor with initial wealth $X_0 = 1$, and assume $C = 3R$. The investor's goal is to maximize his expected utility of time-1 wealth $\mathbb{E}U[R + \pi(S_1 - R)]$, where $U(x) = \sqrt{x}$. Show that the optimal number of shares of stock to hold is $\pi^* = 1$.

What would be the investor's marginal utility price of the contingent claim described above?

30E Partial Differential Equations

Consider the Schrödinger equation

$$i\partial_t \Psi = -\frac{1}{2}\Delta\Psi\,, \qquad x \in \mathbb{R}^n\,,\, t > 0\,,$$

for complex-valued solutions $\Psi(x,t)$ and where Δ is the Laplacian.

(a) Derive, by using a Fourier transform and its inversion, the fundamental solution of the Schrödinger equation. Obtain the solution of the initial value problem

$$\begin{aligned} i\partial_t \Psi &= -\frac{1}{2}\Delta\Psi\,, \qquad x \in \mathbb{R}^n, \quad t > 0\,, \\ \Psi(x,0) &= f(x)\,, \qquad x \in \mathbb{R}^n\,, \end{aligned}$$

as a convolution.

(b) Consider the Wigner-transform of the solution of the Schrödinger equation

$$w(x,\xi,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \Psi(x + \tfrac{1}{2}y,\, t)\, \bar{\Psi}(x - \tfrac{1}{2}y,\, t)\, e^{-iy\cdot\xi}\, \mathrm{d}^n y\,,$$

defined for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $t > 0$. Derive an evolution equation for w by using the Schrödinger equation. Write down the solution of this evolution equation for given initial data $w(x,\xi,0) = g(x,\xi)$.

31C Asymptotic Methods

Consider the ordinary differential equation

$$y'' = (|x| - E)\, y\,,$$

subject to the boundary conditions $y(\pm\infty) = 0$. Write down the general form of the Liouville-Green solutions for this problem for $E > 0$ and show that asymptotically the eigenvalues E_n, $n \in \mathbb{N}$ and $E_n < E_{n+1}$, behave as $E_n = \mathrm{O}(n^{2/3})$ for large n.

32E Integrable Systems

Consider a vector field

$$V = \alpha\, x \frac{\partial}{\partial x} + \beta\, t \frac{\partial}{\partial t} + \gamma\, v \frac{\partial}{\partial v}\,,$$

on $\mathbb{R}^3$, where α, β and γ are constants. Find the one-parameter group of transformations generated by this vector field.

Find the values of the constants (α, β, γ) such that V generates a Lie point symmetry of the modified KdV equation (mKdV)

$$v_t - 6\, v^2\, v_x + v_{xxx} = 0\,, \qquad \text{where} \quad v = v(x,t)\,.$$

Show that the function $u = u(x,t)$ given by $u = v^2 + v_x$ satisfies the KdV equation and find a Lie point symmetry of KdV corresponding to the Lie point symmetry of mKdV which you have determined from V.

33C Principles of Quantum Mechanics

What are the commutation relations between the position operator $\hat{x}$ and momentum operator $\hat{p}$? Show that this is consistent with $\hat{x}$, $\hat{p}$ being hermitian.

The annihilation operator for a harmonic oscillator is

$$a = \sqrt{\frac{1}{2\hbar}}\,(\hat{x} + i\hat{p})$$

in units where the mass and frequency of the oscillator are 1. Derive the relation $[a, a^\dagger] = 1$. Write down an expression for the Hamiltonian

$$H = \tfrac{1}{2}\,\hat{p}^2 + \tfrac{1}{2}\,\hat{x}^2$$

in terms of the operator $N = a^\dagger a$.

Assume there exists a unique ground state $|0\rangle$ of H such that $a|0\rangle = 0$. Explain how the space of eigenstates $|n\rangle$, is formed, and deduce the energy eigenvalues for these states. Show that

$$a|n\rangle = A|n{-}1\rangle\,, \qquad a^\dagger|n\rangle = B|n{+}1\rangle\,,$$

finding A and B in terms of n.

Calculate the energy eigenvalues of the Hamiltonian for two harmonic oscillators

$$H = H_1 + H_2\,, \qquad H_i = \tfrac{1}{2}\,\hat{p}_i^{\,2} + \tfrac{1}{2}\,\hat{x}_i^{\,2}\,, \quad i = 1, 2\,.$$

What is the degeneracy of the n^{th} energy level? Suppose that the two oscillators are then coupled by adding the extra term

$$\Delta H = \lambda \hat{x}_1\,\hat{x}_2$$

to H, where $\lambda \ll 1$. Calculate the energies for the states of the unperturbed system with the three lowest energy eigenvalues to first order in λ using perturbation theory.

[You may assume standard perturbation theory results.]

34B Applications of Quantum Mechanics

State Bloch's theorem for a one dimensional lattice which is invariant under translations by a.

A simple model of a crystal consists of a one-dimensional linear array of identical sites with separation a. At the nth site the Hamiltonian, neglecting all other sites, is H_n and an electron may occupy either of two states, $\phi_n(x)$ and $\chi_n(x)$, where

$$H_n\,\phi_n(x) \;=\; E_0\,\phi_n(x)\,, \qquad\qquad H_n\,\chi_n(x) \;=\; E_1\,\chi_n(x)\,,$$

and ϕ_n and χ_n are orthonormal. How are $\phi_n(x)$ and $\chi_n(x)$ related to $\phi_0(x)$ and $\chi_0(x)$?

The full Hamiltonian is H and is invariant under translations by a. Write trial wavefunctions $\psi(x)$ for the eigenstates of this model appropriate to a tight binding approximation if the electron has probability amplitudes b_n and c_n to be in the states ϕ_n and χ_n respectively.

Assume that the only non-zero matrix elements in this model are, for all n,

$$(\phi_n, H_n\,\phi_n) \;=\; E_0\,, \qquad\qquad (\chi_n, H_n\,\chi_n) \;=\; E_1\,,$$

$$(\phi_n, V\phi_{n\pm1}) \;=\; (\chi_n, V\,\chi_{n\pm1}) \;=\; (\phi_n, V\,\chi_{n\pm1}) \;=\; (\chi_n, V\phi_{n\pm1}) \;=\; -A\,,$$

where $H = H_n + V$ and $A > 0$. Show that the time-dependent Schrödinger equation governing the amplitudes becomes

$$i\hbar\,\dot{b}_n \;=\; E_0\,b_n - A(b_{n+1} + b_{n-1} + c_{n+1} + c_{n-1})\,,$$

$$i\hbar\,\dot{c}_n \;=\; E_1\,c_n - A(c_{n+1} + c_{n-1} + b_{n+1} + b_{n-1})\,.$$

By examining solutions of the form

$$\begin{pmatrix} b_n \\ c_n \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix} e^{\,i(kna - Et/\hbar)}\,,$$

show that the allowed energies of the electron are two bands given by

$$E \;=\; \frac{1}{2}\,(E_0 + E_1 - 4\,A\cos ka) \pm \frac{1}{2}\,\sqrt{(E_0 - E_1)^2 + 16\,A^2\cos^2 ka}\,.$$

Define the *Brillouin zone* for this system and find the energies at the top and bottom of both bands. Hence, show that the energy gap between the bands is

$$\Delta E \;=\; -4\,A + \sqrt{(E_1 - E_0)^2 + 16\,A^2}\,.$$

Show that the wavefunctions $\psi(x)$ satisfy Bloch's theorem.

Describe briefly what are the crucial differences between insulators, conductors and semiconductors.

35C Statistical Physics

(i) Given the following density of states for a particle in 3 dimensions

$$g(\varepsilon) \;=\; KV\varepsilon^{1/2}$$

write down the partition function for a gas of N such non-interacting particles, assuming they can be treated classically. From this expression, calculate the energy E of the system and the heat capacities C_V and C_P. You may take it as given that $PV = \frac{2}{3}E$.

[*Hint: The formula* $\int_0^\infty dy\, y^2\, e^{-y^2} = \sqrt{\pi}/4$ *may be useful.*]

(ii) Using thermodynamic relations obtain the relation between heat capacities and compressibilities

$$\frac{C_P}{C_V} \;=\; \frac{\kappa_T}{\kappa_S}$$

where the isothermal and adiabatic compressibilities are given by

$$\kappa \;=\; -\frac{1}{V}\frac{\partial V}{\partial P}\,,$$

derivatives taken at constant temperature and entropy, respectively.

(iii) Find κ_T and κ_S for the ideal gas considered above.

36B Electrodynamics

A particle of rest-mass m, electric charge q, is moving relativistically along the path $x^\mu(s)$ where s parametrises the path.

Write down an action for which the extremum determines the particle's equation of motion in an electromagnetic field given by the potential $A^\mu(x)$.

Use your action to derive the particle's equation of motion in a form where s is the proper time.

Suppose that the electric and magnetic fields are given by

$$\begin{aligned}\mathbf{E} &= (0, 0, E)\,,\\ \mathbf{B} &= (0, B, 0)\,.\end{aligned}$$

where E and B are constants and $B > E > 0$.

Find $x^\mu(s)$ given that the particle starts at rest at the origin when $s = 0$.

Describe qualitatively the motion of the particle.

37A Fluid Dynamics II

The equation for the vorticity $\omega(x,y)$ in two-dimensional incompressible flow takes the form

$$\frac{\partial \omega}{\partial t} + u\,\frac{\partial \omega}{\partial x} + v\,\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right),$$

where

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad \omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right),$$

and $\psi(x,y)$ is the stream function.

Show that this equation has a time-dependent similarity solution of the form

$$\psi = CxH(t)^{-1}\phi(\eta), \qquad \omega = -CxH(t)^{-3}\phi_{\eta\eta}(\eta) \quad \text{for} \quad \eta = yH(t)^{-1},$$

if $H(t) = \sqrt{2Ct}$ and ϕ satisfies the equation

$$3\,\phi_{\eta\eta} + \eta\phi_{\eta\eta\eta} - \phi_\eta\phi_{\eta\eta} + \phi\phi_{\eta\eta\eta} + \frac{1}{R}\,\phi_{\eta\eta\eta\eta} = 0, \qquad (*)$$

and $R = C/\nu$ is the effective Reynolds number.

Show that this solution is appropriate for the problem of two-dimensional flow between the rigid planes $y = \pm H(t)$, and determine the boundary conditions on ϕ in that case.

Verify that $(*)$ has exact solutions, satisfying the boundary conditions, of the form

$$\phi = \frac{(-1)^k}{k\pi}\sin(k\pi\eta) - \eta, \qquad k = 1, 2, \ldots\ ,$$

when $R = k^2\pi^2/4$. Sketch this solution when k is large, and discuss whether such solutions are likely to be realised in practice.

38A Waves

Consider the equation

$$\frac{\partial^2 \phi}{\partial t \partial x} = -\alpha \, \phi \, ,$$

where α is a positive constant. Find the dispersion relation for waves of frequency ω and wavenumber k. Sketch graphs of the phase velocity $c(k)$ and the group velocity $c_g(k)$.

A disturbance localized near $x = 0$ at $t = 0$ evolves into a dispersing wave packet. Will the wavelength and frequency of the waves passing a stationary observer located at a large positive value of x increase or decrease for $t > 0$? In which direction do the crests pass the observer?

Write down the solution $\phi(x, t)$ with initial value

$$\phi(x, 0) = \int_{-\infty}^{\infty} A(k) \, e^{ikx} \, dk \, .$$

What can be said about $A(-k)$ if ϕ is real?

Use the method of stationary phase to obtain an approximation for $\phi(Vt, t)$ for fixed $V > 0$ and large t. What can be said about the solution at $x = -Vt$ for large t?

[You may assume that $\displaystyle\int_{-\infty}^{\infty} e^{-au^2} \, du = \sqrt{\frac{\pi}{a}}$ for $\mathrm{Re}(a) \geqslant 0$, $a \neq 0$.]

39A Numerical Analysis

The Poisson equation $\nabla^2 u = f$ in the unit square $\Omega = [0,1] \times [0,1]$, $u = 0$ on $\partial \Omega$, is discretized with the five-point formula

$$u_{i,j-1} + u_{i,j+1} + u_{i+1,j} + u_{i-1,j} - 4 \, u_{i,j} = h^2 f_{i,j} \, ,$$

where $1 \leqslant i, \; j \leqslant M$, $u_{i,j} \approx u(ih, jh)$ and (ih, jh) are grid points.

Let $u(x, y)$ be the exact solution, and let $e_{i,j} = u_{i,j} - u(ih, jh)$ be the error of the five-point formula at the (i, j)th grid point. Justifying each step, prove that

$$\|\mathbf{e}\| = \left[\sum_{i,j=1}^{M} | \, e_{i,j} |^2 \right]^{1/2} \leqslant ch \quad \text{for sufficiently small} \quad h > 0 \, ,$$

where c is some constant independent of h.

END OF PAPER