UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 3 June, 2010 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Groups Rings and Modules

Let A be the ring of integers $\mathbb{Z}$ or the polynomial ring $\mathbb{C}[X]$. In each case, give an example of an ideal I of A such that the quotient ring $R = A/I$ has a non-trivial idempotent (an element $x \in R$ with $x \neq 0, 1$ and $x^2 = x$) and a non-trivial nilpotent element (an element $x \in R$ with $x \neq 0$ and $x^n = 0$ for some positive integer n). Exhibit these elements and justify your answer.

2G Analysis II

Consider the map $f : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$f(x,\, y,\, z) \;=\; (x + y + z,\; xy + yz + zx,\; xyz)\,.$$

Show that f is differentiable everywhere and find its derivative.

Stating carefully any theorem that you quote, show that f is locally invertible near a point (x, y, z) unless $(x - y)(y - z)(z - x) = 0$.

3H Metric and Topological Spaces

Let X be a topological space and Y be a set. Let $p : X \to Y$ be a surjection. The *quotient topology* on Y is defined as follows: a subset $V \subset Y$ is open if and only if $p^{-1}(V)$ is open in X.

(1) Show that this does indeed define a topology on Y, and show that p is continuous when we endow Y with this topology.

(2) Let Z be another topological space and $f : Y \to Z$ be a map. Show that f is continuous if and only if $f \circ p : X \to Z$ is continuous.

4A Complex Methods

(a) Prove that the real and imaginary parts of a complex differentiable function are harmonic.

(b) Find the most general harmonic polynomial of the form

$$u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

where a, b, c, d, x and y are real.

(c) Write down a complex analytic function of $z = x + iy$ of which $u(x, y)$ is the real part.

5F Geometry

(i) Write down the Poincaré metric on the unit disc model D of the hyperbolic plane. Compute the hyperbolic distance ρ from $(0, 0)$ to $(r, 0)$, with $0 < r < 1$.

(ii) Given a point P in D and a hyperbolic line L in D with P not on L, describe how the minimum distance from P to L is calculated. Justify your answer.

6D Variational Principles

Derive the Euler-Lagrange equation for the function $u(x, y)$ which gives a stationary value of

$$I = \int_{\mathcal{D}} L\Big(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)\, dxdy,$$

where $\mathcal{D}$ is a bounded domain in the (x, y) plane, with u fixed on the boundary $\partial\mathcal{D}$.

Find the equation satisfied by the function u which gives a stationary value of

$$I = \int_{\mathcal{D}} \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\Big]\, dxdy,$$

with u given on $\partial\mathcal{D}$.

7B Methods

Show that Laplace's equation $\nabla^2\phi = 0$ in polar coordinates (r,θ) has solutions proportional to $r^{\pm\alpha}\sin\alpha\theta$, $r^{\pm\alpha}\cos\alpha\theta$ for any constant α.

Find the function ϕ satisfying Laplace's equation in the region $a < r < b$, $0 < \theta < \pi$, where $\phi(a,\theta) = \sin^3\theta$, $\phi(b,\theta) = \phi(r,0) = \phi(r,\pi) = 0$.

[*The Laplacian* ∇^2 *in polar coordinates is*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.\Big]$$

8D Quantum Mechanics

Write down the commutation relations between the components of position $\mathbf{x}$ and momentum $\mathbf{p}$ for a particle in three dimensions.

A particle of mass m executes simple harmonic motion with Hamiltonian

$$H = \frac{1}{2m}\mathbf{p}^2 + \frac{m\omega^2}{2}\mathbf{x}^2,$$

and the orbital angular momentum operator is defined by

$$\mathbf{L} = \mathbf{x}\times\mathbf{p}.$$

Show that the components of $\mathbf{L}$ are observables commuting with H. Explain briefly why the components of $\mathbf{L}$ are not simultaneous observables. What are the implications for the labelling of states of the three-dimensional harmonic oscillator?

9E Markov Chains

An intrepid tourist tries to ascend Springfield's famous infinite staircase on an icy day. When he takes a step with his right foot, he reaches the next stair with probability $1/2$, otherwise he falls down and instantly slides back to the bottom with probability $1/2$. Similarly, when he steps with his left foot, he reaches the next stair with probability $1/3$, or slides to the bottom with probability $2/3$. Assume that he always steps first with his right foot when he is at the bottom, and alternates feet as he ascends. Let X_n be his position after his nth step, so that $X_n = i$ when he is on the stair i, $i = 0, 1, 2, \ldots$, where 0 is the bottom stair.

(a) Specify the transition probabilities p_{ij} for the Markov chain $(X_n)_{n \geqslant 0}$ for any $i, j \geqslant 0$.

(b) Find the equilibrium probabilities π_i, for $i \geqslant 0$. [Hint: $\pi_0 = 5/9$.]

(c) Argue that the chain is irreducible and aperiodic and evaluate the limit

$$\lim_{n \to \infty} \mathbb{P}(X_n = i)$$

for each $i \geqslant 0$.

SECTION II

10F Linear Algebra

Suppose that V is a finite-dimensional vector space over $\mathbb{C}$, and that $\alpha : V \to V$ is a $\mathbb{C}$-linear map such that $\alpha^n = 1$ for some $n > 1$. Show that if V_1 is a subspace of V such that $\alpha(V_1) \subset V_1$, then there is a subspace V_2 of V such that $V = V_1 \oplus V_2$ and $\alpha(V_2) \subset V_2$.

[Hint: Show, for example by picking bases, that there is a linear map $\pi : V \to V_1$ with $\pi(x) = x$ for all $x \in V_1$. Then consider $\rho : V \to V_1$ with $\rho(y) = \frac{1}{n} \sum_{i=0}^{n-1} \alpha^i \pi \alpha^{-i}(y)$.]

11H Groups Rings and Modules

Let R be an integral domain and $R^\times$ its group of units. An element of $S = R \setminus (R^\times \cup \{0\})$ is *irreducible* if it is not a product of two elements in S. When R is Noetherian, show that every element of S is a product of finitely many irreducible elements of S.

12G Analysis II

Let $f : U \to \mathbf{R}^n$ be a map on an open subset $U \subset \mathbf{R}^m$. Explain what it means for f to be differentiable on U. If $g : V \to \mathbf{R}^m$ is a differentiable map on an open subset $V \subset \mathbf{R}^p$ with $g(V) \subset U$, state and prove the Chain Rule for the derivative of the composite fg.

Suppose now $F : \mathbf{R}^n \to \mathbf{R}$ is a differentiable function for which the partial derivatives $D_1F(\mathbf{x}) = D_2F(\mathbf{x}) = \ldots = D_nF(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^n$. By considering the function $G : \mathbf{R}^n \to \mathbf{R}$ given by

$$G(y_1, \ldots, y_n) = F\left(y_1, \ldots, y_{n-1}, y_n - \sum_{i=1}^{n-1} y_i\right),$$

or otherwise, show that there exists a differentiable function $h : \mathbf{R} \to \mathbf{R}$ with $F(x_1, \ldots, x_n) = h(x_1 + \cdots + x_n)$ at all points of $\mathbf{R}^n$.

13G Complex Analysis

State Morera's theorem. Suppose f_n $(n = 1, 2, \ldots)$ are analytic functions on a domain $U \subset \mathbf{C}$ and that f_n tends locally uniformly to f on U. Show that f is analytic on U. Explain briefly why the derivatives f_n' tend locally uniformly to f'.

Suppose now that the f_n are nowhere vanishing and f is not identically zero. Let a be any point of U; show that there exists a closed disc $\overline{\Delta} \subset U$ with centre a, on which the convergence of f_n and f_n' are both uniform, and where f is nowhere zero on $\overline{\Delta} \setminus \{a\}$. By considering

$$\frac{1}{2\pi i} \int_C \frac{f_n'(w)}{f_n(w)}\, dw$$

(where C denotes the boundary of $\overline{\Delta}$), or otherwise, deduce that $f(a) \neq 0$.

14F Geometry

Describe the hyperbolic metric on the upper half-plane H. Show that any Möbius transformation that preserves H is an isometry of this metric.

Suppose that $z_1, z_2 \in H$ are distinct and that the hyperbolic line through z_1 and z_2 meets the real axis at w_1, w_2. Show that the hyperbolic distance $\rho(z_1, z_2)$ between z_1 and z_2 is given by $\rho(z_1, z_2) = \log r$, where r is the cross-ratio of the four points z_1, z_2, w_1, w_2, taken in an appropriate order.

15A Methods

(a) Put the equation

$$x\frac{d^2u}{dx^2}+\frac{du}{dx}+\lambda xu=0, \qquad 0\leqslant x\leqslant 1,$$

into Sturm–Liouville form.

(b) Suppose $u_n(x)$ are eigenfunctions such that $u_n(x)$ are bounded as x tends to zero and

$$x\frac{d^2u_n}{dx^2}+\frac{du_n}{dx}+\lambda_n xu_n=0, \qquad 0\leqslant x\leqslant 1.$$

Identify the weight function $w(x)$ and the most general boundary conditions on $u_n(x)$ which give the orthogonality relation

$$(\lambda_m-\lambda_n)\int_0^1 u_m(x)w(x)u_n(x)\,dx=0.$$

(c) The equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0, \qquad x>0,$$

has a solution $J_0(x)$ and a second solution which is not bounded at the origin. The zeros of $J_0(x)$ arranged in ascending order are $j_n, n=1,2,....$ Given that $u_n(1)=0$, show that the eigenvalues of the Sturm–Liouville problem in (b) are $\lambda={j_n}^2, n=1,2,....$

(d) Using the differential equations for $J_0(\alpha x)$ and $J_0(\beta x)$ and integration by parts, show that

$$\int_0^1 J_0(\alpha x)J_0(\beta x)x\,dx=\frac{\beta J_0(\alpha)J_0'(\beta)-\alpha J_0(\beta)J_0'(\alpha)}{\alpha^2-\beta^2} \qquad (\alpha\neq\beta).$$

16D Quantum Mechanics

A π^- (a particle of the same charge as the electron but 270 times more massive) is bound in the Coulomb potential of a proton. Assuming that the wave function has the form $ce^{-r/a}$, where c and a are constants, determine the normalized wave function of the lowest energy state of the π^-, assuming it to be an S-wave (i.e. the state with $l = 0$). (You should treat the proton as fixed in space.)

Calculate the probability of finding the π^- inside a sphere of radius R in terms of the ratio $\mu = R/a$, and show that this probability is given by $4\mu^3/3 + O(\mu^4)$ if μ is very small. Would the result be larger or smaller if the π^- were in a P-wave ($l = 1$) state? Justify your answer very briefly.

[Hint: in spherical polar coordinates,

$$\nabla^2\psi(\mathbf{r}) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}.\Big]$$

17C Electromagnetism

Write down Maxwell's equations in a region with no charges and no currents. Show that if $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ is a solution then so is $\widetilde{\mathbf{E}}(\mathbf{x},t) = c\mathbf{B}(\mathbf{x},t)$ and $\widetilde{\mathbf{B}}(\mathbf{x},t) = -\mathbf{E}(\mathbf{x},t)/c$. Write down the boundary conditions on $\mathbf{E}$ and $\mathbf{B}$ at the boundary with unit normal $\mathbf{n}$ between a perfect conductor and a vacuum.

Electromagnetic waves propagate inside a tube of perfectly conducting material. The tube's axis is in the z-direction, and it is surrounded by a vacuum. The fields may be taken to be the real parts of

$$\mathbf{E}(\mathbf{x},t) = \mathbf{e}(x,y)e^{i(kz-\omega t)}, \qquad \mathbf{B}(\mathbf{x},t) = \mathbf{b}(x,y)e^{i(kz-\omega t)}\,.$$

Write down Maxwell's equations in terms of $\mathbf{e}$, $\mathbf{b}$, k and ω.

Suppose first that $b_z(x,y) = 0$. Show that the solution is determined by

$$\mathbf{e} = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, ik\left[1 - \frac{\omega^2}{k^2c^2}\right]\psi\right),$$

where the function $\psi(x,y)$ satisfies

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \gamma^2\psi = 0\,,$$

and ψ vanishes on the boundary of the tube. Here γ^2 is a constant whose value should be determined.

Obtain a similar condition for the case where $e_z(x,y) = 0$. [*You may find it useful to use a result from the first paragraph.*] What is the corresponding boundary condition?

18B Fluid Dynamics

Write down the exact kinematic and dynamic boundary conditions that apply at the free surface $z = \eta(x,t)$ of a fluid layer in the presence of gravity in the z-direction. Show how these may be approximated for small disturbances of a hydrostatic state about $z = 0$. (The flow of the fluid is in the (x,z)-plane and may be taken to be irrotational, and the pressure at the free surface may be assumed to be constant.)

Fluid of density ρ fills the region $0 > z > -h$. At $z = -h$ the z-component of the velocity is $\epsilon\mathrm{Re}(e^{i\omega t}\cos kx)$, where $|\epsilon| \ll 1$. Find the resulting disturbance of the free surface, assuming this to be small. Explain physically why your answer has a singularity for a particular value of ω^2.

19C Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A and explain how it can be used to solve the least squares problem of finding the $x^* \in \mathbf{R}^n$ which minimises $\|Ax - b\|$ where $b \in \mathbf{R}^m$, $m > n$, and the norm is the Euclidean one.

Define a Householder (reflection) transformation H and show that it is an orthogonal matrix.

Using a Householder reflection, solve the least squares problem for

$$A = \begin{pmatrix} 2 & 4 & 7 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 9 \\ -7 \\ 3 \\ 1 \\ -1 \end{pmatrix},$$

giving both x^* and $\|Ax^* - b\|$.

20E Statistics

Let $X_1, \ldots, X_n$ be independent $\mathrm{Exp}(\theta)$ random variables with unknown parameter θ. Find the maximum likelihood estimator $\hat{\theta}$ of θ, and state the distribution of $n/\hat{\theta}$. Show that $\theta/\hat{\theta}$ has the $\Gamma(n, n)$ distribution. Find the $100\,(1-\alpha)\%$ confidence interval for θ of the form $[0, C\,\hat{\theta}]$ for a constant $C > 0$ depending on α.

Now, taking a Bayesian point of view, suppose your prior distribution for the parameter θ is $\Gamma(k, \lambda)$. Show that your Bayesian point estimator $\hat{\theta}_B$ of θ for the loss function $L(\theta, a) = (\theta - a)^2$ is given by

$$\hat{\theta}_B = \frac{n + k}{\lambda + \sum_i X_i}.$$

Find a constant $C_B > 0$ depending on α such that the posterior probability that $\theta \leqslant C_B\,\hat{\theta}_B$ is equal to $1 - \alpha$.

[*The density of the* $\Gamma(k, \lambda)$ *distribution is* $f(x;\, k, \lambda) = \lambda^k x^{k-1} e^{-\lambda x}/\Gamma(k)$, *for* $x > 0$.]

21E Optimization

Let A be the $m \times n$ payoff matrix of a two-person, zero-sum game. What is Player I's optimization problem?

Write down a sufficient condition that a vector $p \in \mathbb{R}^m$ is an optimal mixed strategy for Player I in terms of the optimal mixed strategy of Player II and the value of the game. If $m = n$ and A is an invertible, symmetric matrix such that $A^{-1}e \geqslant 0$, where $e = (1, \ldots, 1)^T \in \mathbb{R}^m$, show that the value of the game is $(e^T A^{-1} e)^{-1}$.

Consider the following game: Players I and II each have three cards labelled $1, 2$, and 3. Each player chooses one of her cards, independently of the other, and places it in the same envelope. If the sum of the numbers in the envelope is smaller than or equal to 4, then Player II pays Player I the sum (in £), and otherwise Player I pays Player II the sum. (For instance, if Player I chooses card 3 and Player II choose card 2, then Player I pays Player II £5.) What is the optimal strategy for each player?

END OF PAPER