UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday, 2 June, 2010 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Suppose that ϕ is an endomorphism of a finite-dimensional complex vector space.

(i) Show that if λ is an eigenvalue of ϕ, then λ^2 is an eigenvalue of ϕ^2.

(ii) Show conversely that if μ is an eigenvalue of ϕ^2, then there is an eigenvalue λ of ϕ with $\lambda^2 = \mu$.

2H Groups Rings and Modules

Give the definition of conjugacy classes in a group G. How many conjugacy classes are there in the symmetric group S_4 on four letters? Briefly justify your answer.

3G Analysis II

Let $c > 1$ be a real number, and let F_c be the space of sequences $\mathbf{a} = (a_1, a_2, \dots)$ of real numbers a_i with $\sum_{r=1}^{\infty} c^{-r}|a_r|$ convergent. Show that $\|\mathbf{a}\|_c = \sum_{r=1}^{\infty} c^{-r}|a_r|$ defines a norm on F_c.

Let F denote the space of sequences $\mathbf{a}$ with $|a_i|$ bounded; show that $F \subset F_c$. If $c' > c$, show that the norms on F given by restricting to F the norms $\|\,.\,\|_c$ on F_c and $\|\,.\,\|_{c'}$ on $F_{c'}$ are not Lipschitz equivalent.

By considering sequences of the form $\mathbf{a}^{(n)} = (a,\, a^2,\, \dots\,,\, a^n,\, 0,\, 0,\, \dots)$ in F, for a an appropriate real number, or otherwise, show that F (equipped with the norm $\|\,.\,\|_c$) is not complete.

4H Metric and Topological Spaces

On the set $\mathbb{Q}$ of rational numbers, the 3-*adic metric* d_3 is defined as follows: for $x, y \in \mathbb{Q}$, define $d_3(x,x) = 0$ and $d_3(x,y) = 3^{-n}$, where n is the integer satisfying $x - y = 3^n u$ where u is a rational number whose denominator and numerator are both prime to 3.

(1) Show that this is indeed a metric on $\mathbb{Q}$.

(2) Show that in $(\mathbb{Q}, d_3)$, we have $3^n \to 0$ as $n \to \infty$ while $3^{-n} \not\to 0$ as $n \to \infty$. Let d be the usual metric $d(x,y) = |x - y|$ on $\mathbb{Q}$. Show that neither the identity map $(\mathbb{Q}, d) \to (\mathbb{Q}, d_3)$ nor its inverse is continuous.

5A Methods

Consider the initial value problem

$$\mathcal{L}x(t) = f(t), \quad x(0) = 0, \quad \dot{x}(0) = 0, \quad t \geqslant 0,$$

where $\mathcal{L}$ is a second-order linear operator involving differentiation with respect to t. Explain briefly how to solve this by using a Green's function.

Now consider

$$\ddot{x}(t) = \begin{cases} a & 0 \leqslant t \leqslant T, \\ 0 & T < t < \infty, \end{cases}$$

where a is a constant, subject to the same initial conditions. Solve this using the Green's function, and explain how your answer is related to a problem in Newtonian dynamics.

6C Electromagnetism

Write down Maxwell's equations for electromagnetic fields in a non-polarisable and non-magnetisable medium.

Show that the homogenous equations (those not involving charge or current densities) can be solved in terms of vector and scalar potentials $\mathbf{A}$ and ϕ.

Then re-express the inhomogeneous equations in terms of $\mathbf{A}$, ϕ and $f = \boldsymbol{\nabla}\cdot\mathbf{A} + c^{-2}\dot{\phi}$. Show that the potentials can be chosen so as to set $f = 0$ and hence rewrite the inhomogeneous equations as wave equations for the potentials. [*You may assume that the inhomogeneous wave equation* $\nabla^2\psi - c^{-2}\ddot{\psi} = \sigma(\mathbf{x}, t)$ *always has a solution* $\psi(\mathbf{x}, t)$ *for any given* $\sigma(\mathbf{x}, t)$.]

7B Fluid Dynamics

Write down an expression for the velocity field of a line vortex of strength κ.

Consider N identical line vortices of strength κ arranged at equal intervals round a circle of radius a. Show that the vortices all move around the circle at constant angular velocity $(N-1)\kappa/(4\pi a^2)$.

8E Statistics

A washing powder manufacturer wants to determine the effectiveness of a television advertisement. Before the advertisement is shown, a pollster asks 100 randomly chosen people which of the three most popular washing powders, labelled A, B and C, they prefer. After the advertisement is shown, another 100 randomly chosen people (not the same as before) are asked the same question. The results are summarized below.

	A	B	C
before	36	47	17
after	44	33	23

Derive and carry out an appropriate test at the 5% significance level of the hypothesis that the advertisement has had no effect on people's preferences.

[*You may find the following table helpful:*

	χ_1^2	χ_2^2	χ_3^2	χ_4^2	χ_5^2	χ_6^2
95 percentile	3.84	5.99	7.82	9.49	11.07	12.59

.]

9E Optimization

Consider the function ϕ defined by

$$\phi(b) = \inf\{x^2 + y^4 : x + 2y = b\}.$$

Use the Lagrangian sufficiency theorem to evaluate $\phi(3)$. Compute the derivative $\phi'(3)$.

SECTION II

10F Linear Algebra

(i) Show that two $n \times n$ complex matrices A, B are similar (i.e. there exists invertible P with $A = P^{-1}BP$) if and only if they represent the same linear map $\mathbb{C}^n \to \mathbb{C}^n$ with respect to different bases.

(ii) Explain the notion of Jordan normal form of a square complex matrix.

(iii) Show that any square complex matrix A is similar to its transpose.

(iv) If A is invertible, describe the Jordan normal form of A^{-1} in terms of that of A.

Justify your answers.

11H Groups Rings and Modules

For ideals I, J of a ring R, their *product* IJ is defined as the ideal of R generated by the elements of the form xy where $x \in I$ and $y \in J$.

(1) Prove that, if a prime ideal P of R contains IJ, then P contains either I or J.

(2) Give an example of R, I and J such that the two ideals IJ and $I \cap J$ are different from each other.

(3) Prove that there is a natural bijection between the prime ideals of R/IJ and the prime ideals of $R/(I \cap J)$.

12G Analysis II

Suppose the functions f_n $(n = 1, 2, \ldots)$ are defined on the open interval $(0, 1)$ and that f_n tends uniformly on $(0, 1)$ to a function f. If the f_n are continuous, show that f is continuous. If the f_n are differentiable, show by example that f need not be differentiable.

Assume now that each f_n is differentiable and the derivatives f_n' converge uniformly on $(0, 1)$. For any given $c \in (0, 1)$, we define functions $g_{c,n}$ by

$$g_{c,n}(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x - c} & \text{for } x \neq c\,, \\ f_n'(c) & \text{for } x = c\,. \end{cases}$$

Show that each $g_{c,n}$ is continuous. Using the general principle of uniform convergence (the Cauchy criterion) and the Mean Value Theorem, or otherwise, prove that the functions $g_{c,n}$ converge uniformly to a continuous function g_c on $(0, 1)$, where

$$g_c(x) = \frac{f(x) - f(c)}{x - c} \qquad \text{for } x \neq c\ .$$

Deduce that f is differentiable on $(0, 1)$.

13A Complex Analysis or Complex Methods

(a) Prove that a complex differentiable map, $f(z)$, is conformal, i.e. preserves angles, provided a certain condition holds on the first complex derivative of $f(z)$.

(b) Let D be the region

$$D := \{z \in \mathbb{C} : |z-1| > 1 \text{ and } |z-2| < 2\}.$$

Draw the region D. It might help to consider the two sets

$$C(1) := \{z \in \mathbb{C} : |z - 1| = 1\},$$
$$C(2) := \{z \in \mathbb{C} : |z - 2| = 2\}.$$

(c) For the transformations below identify the images of D.

Step 1: The first map is $f_1(z) = \dfrac{z-1}{z}$,

Step 2: The second map is the composite $f_2 f_1$ where $f_2(z) = (z - \frac{1}{2})i$,

Step 3: The third map is the composite $f_3 f_2 f_1$ where $f_3(z) = e^{2\pi z}$.

(d) Write down the inverse map to the composite $f_3 f_2 f_1$, explaining any choices of branch.

[*The composite* $f_2 f_1$ *means* $f_2(f_1(z))$.]

14F Geometry

Suppose that $a > 0$ and that $S \subset \mathbb{R}^3$ is the half-cone defined by $z^2 = a(x^2 + y^2)$, $z > 0$. By using an explicit smooth parametrization of S, calculate the curvature of S.

Describe the geodesics on S. Show that for $a = 3$, no geodesic intersects itself, while for $a > 3$ some geodesic does so.

15D Variational Principles

Describe briefly the method of Lagrange multipliers for finding the stationary points of a function $f(x, y)$ subject to a constraint $\phi(x, y) = 0$.

A tent manufacturer wants to maximize the volume of a new design of tent, subject only to a constant weight (which is directly proportional to the amount of fabric used). The models considered have either equilateral-triangular or semi-circular vertical cross–section, with vertical planar ends in both cases and with floors of the same fabric. Which shape maximizes the volume for a given area A of fabric?

[Hint: $(2\pi)^{-1/2} 3^{-3/4} (2 + \pi) < 1$.]

16B Methods

Explain briefly the use of the method of characteristics to solve linear first-order partial differential equations.

Use the method to solve the problem

$$(x - y)\frac{\partial u}{\partial x} + (x + y)\frac{\partial u}{\partial y} = \alpha u,$$

where α is a constant, with initial condition $u(x, 0) = x^2$, $x \geqslant 0$.

By considering your solution explain:

(i) why initial conditions cannot be specified on the whole x-axis;

(ii) why a single-valued solution in the entire plane is not possible if $\alpha \neq 2$.

17D Quantum Mechanics

A particle of mass m moves in a one-dimensional potential defined by

$$V(x) = \begin{cases} \infty & \text{for } x < 0, \\ 0 & \text{for } 0 \leqslant x \leqslant a, \\ V_0 & \text{for } a < x, \end{cases}$$

where a and V_0 are positive constants. Defining $c = [2m(V_0 - E)]^{1/2}/\hbar$ and $k = (2mE)^{1/2}/\hbar$, show that for any allowed positive value E of the energy with $E < V_0$ then

$$c + k \cot ka = 0.$$

Find the minimum value of V_0 for this equation to have a solution.

Find the normalized wave function for the particle. Write down an expression for the expectation value of x in terms of two integrals, which you need not evaluate. Given that

$$\langle x \rangle = \frac{1}{2k}(ka - \tan ka),$$

discuss briefly the possibility of $\langle x \rangle$ being greater than a. [Hint: consider the graph of $-ka \cot ka$ against ka.]

18C Electromagnetism

A steady current I_2 flows around a loop $\mathcal{C}_2$ of a perfectly conducting narrow wire. Assuming that the gauge condition $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ holds, the vector potential at points away from the loop may be taken to be

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I_2}{4\pi} \oint_{\mathcal{C}_2} \frac{d\mathbf{r}_2}{|\mathbf{r} - \mathbf{r}_2|} .$$

First verify that the gauge condition is satisfied here. Then obtain the Biot-Savart formula for the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I_2}{4\pi} \oint_{\mathcal{C}_2} \frac{d\mathbf{r}_2 \times (\mathbf{r} - \mathbf{r}_2)}{|\mathbf{r} - \mathbf{r}_2|^3} .$$

Next suppose there is a similar but separate loop $\mathcal{C}_1$ with current I_1. Show that the magnetic force exerted on loop $\mathcal{C}_1$ by loop $\mathcal{C}_2$ is

$$\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} d\mathbf{r}_1 \times \left(d\mathbf{r}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right) .$$

Is this consistent with Newton's third law? Justify your answer.

19C Numerical Analysis

Consider the initial value problem for an autonomous differential equation

$$y'(t) = f(y(t)), \qquad y(0) = y_0 \text{ given},$$

and its approximation on a grid of points $t_n = nh$, $n = 0, 1, 2, \ldots$. Writing $y_n = y(t_n)$, it is proposed to use one of two Runge–Kutta schemes defined by

$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2),$$

where $k_1 = hf(y_n)$ and

$$k_2 = \begin{cases} hf(y_n + k_1) & \text{scheme I}, \\ hf(y_n + \frac{1}{2}(k_1 + k_2)) & \text{scheme II}. \end{cases}$$

What is the order of each scheme? Determine the A-stability of each scheme.

20E Markov Chains

Let $(X_n)_{n \geqslant 0}$ be a simple, symmetric random walk on the integers $\{\ldots, -1, 0, 1, \ldots\}$, with $X_0 = 0$ and $\mathbb{P}(X_{n+1} = i \pm 1 \,|\, X_n = i) = 1/2$. For each integer $a \geqslant 1$, let $T_a = \inf\{n \geqslant 0 : X_n = a\}$. Show that T_a is a stopping time.

Define a random variable Y_n by the rule

$$Y_n = \begin{cases} X_n & \text{if } n < T_a, \\ 2a - X_n & \text{if } n \geqslant T_a. \end{cases}$$

Show that $(Y_n)_{n \geqslant 0}$ is also a simple, symmetric random walk.

Let $M_n = \max_{0 \leqslant i \leqslant n} X_n$. Explain why $\{M_n \geqslant a\} = \{T_a \leqslant n\}$ for $a \geqslant 0$. By using the process $(Y_n)_{n \geqslant 0}$ constructed above, show that, for $a \geqslant 0$,

$$\mathbb{P}(M_n \geqslant a, X_n \leqslant a - 1) = \mathbb{P}(X_n \geqslant a + 1),$$

and thus

$$\mathbb{P}(M_n \geqslant a) = \mathbb{P}(X_n \geqslant a) + \mathbb{P}(X_n \geqslant a + 1).$$

Hence compute

$$\mathbb{P}(M_n = a)$$

when a and n are positive integers with $n \geqslant a$. [Hint: if n is even, then X_n must be even, and if n is odd, then X_n must be odd.]

END OF PAPER