UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday, 1 June, 2010 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Suppose that V is the complex vector space of polynomials of degree at most $n-1$ in the variable z. Find the Jordan normal form for each of the linear transformations $\frac{d}{dz}$ and $z\frac{d}{dz}$ acting on V.

2A Complex Analysis or Complex Methods

(a) Write down the definition of the complex derivative of the function $f(z)$ of a single complex variable.

(b) Derive the Cauchy-Riemann equations for the real and imaginary parts $u(x, y)$ and $v(x, y)$ of $f(z)$, where $z = x + iy$ and

$$f(z) = u(x, y) + iv(x, y).$$

(c) State necessary and sufficient conditions on $u(x, y)$ and $v(x, y)$ for the function $f(z)$ to be complex differentiable.

3F Geometry

(i) Define the notion of curvature for surfaces embedded in $\mathbb{R}^3$.

(ii) Prove that the unit sphere in $\mathbb{R}^3$ has curvature $+1$ at all points.

4D Variational Principles

(a) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be convex and strictly convex.

(b) State a necessary and sufficient first-order condition for strict convexity of $f \in C^1(\mathbb{R}^n)$, and give, with proof, an example of a function which is strictly convex but with second derivative which is not everywhere strictly positive.

5B Fluid Dynamics

A planar solenoidal velocity field has the velocity potential

$$\phi(x,y,t) = xe^{-t} + ye^{t}.$$

Find and sketch (i) the streamlines at $t=0$; (ii) the pathline that passes through the origin at $t=0$; (iii) the locus at $t=0$ of points that pass through the origin at earlier times (streakline).

6C Numerical Analysis

Obtain the Cholesky decompositions of

$$H_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}, \qquad H_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \lambda \end{pmatrix}.$$

What is the minimum value of λ for H_4 to be positive definite? Verify that if $\lambda = \frac{1}{7}$ then H_4 is positive definite.

7E Statistics

Suppose $X_1, \ldots, X_n$ are independent $N(0,\sigma^2)$ random variables, where σ^2 is an unknown parameter. Explain carefully how to construct the uniformly most powerful test of size α for the hypothesis $H_0 : \sigma^2 = 1$ versus the alternative $H_1 : \sigma^2 > 1$.

8E Optimization

What is the maximal flow problem in a network?

Explain the Ford–Fulkerson algorithm. Why must this algorithm terminate if the initial flow is set to zero and all arc capacities are rational numbers?

SECTION II

9F Linear Algebra

Let V denote the vector space of $n \times n$ real matrices.

(1) Show that if $\psi(A,B) = \operatorname{tr}(AB^T)$, then ψ is a positive-definite symmetric bilinear form on V.

(2) Show that if $q(A) = \operatorname{tr}(A^2)$, then q is a quadratic form on V. Find its rank and signature.

[Hint: Consider symmetric and skew-symmetric matrices.]

10H Groups Rings and Modules

Prove that the kernel of a group homomorphism $f : G \to H$ is a normal subgroup of the group G.

Show that the dihedral group D_8 of order 8 has a non-normal subgroup of order 2. Conclude that, for a group G, a normal subgroup of a normal subgroup of G is not necessarily a normal subgroup of G.

11G Analysis II

State and prove the contraction mapping theorem. Demonstrate its use by showing that the differential equation $f'(x) = f(x^2)$, with boundary condition $f(0) = 1$, has a unique solution on $[0,1)$, with one-sided derivative $f'(0) = 1$ at zero.

12H Metric and Topological Spaces

Let $f : X \to Y$ and $g : Y \to X$ be continuous maps of topological spaces with $f \circ g = \mathrm{id}_Y$.

(1) Suppose that (i) Y is path-connected, and (ii) for every $y \in Y$, its inverse image $f^{-1}(y)$ is path-connected. Prove that X is path-connected.

(2) Prove the same statement when "path-connected" is everywhere replaced by "connected".

13A Complex Analysis or Complex Methods

Calculate the following real integrals by using contour integration. Justify your steps carefully.

(a)

$$I_1 = \int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx, \qquad a > 0,$$

(b)

$$I_2 = \int_0^\infty \frac{x^{1/2} \log x}{1 + x^2}\, dx.$$

14A Methods

(a) A function $f(t)$ is periodic with period 2π and has continuous derivatives up to and including the kth derivative. Show by integrating by parts that the Fourier coefficients of $f(t)$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt\, dt,$$
$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt\, dt,$$

decay at least as fast as $1/n^k$ as $n \to \infty$.

(b) Calculate the Fourier series of $f(t) = |\sin t|$ on $[0, 2\pi]$.

(c) Comment on the decay rate of your Fourier series.

15D Quantum Mechanics

A particle of unit mass moves in one dimension in a potential

$$V = \frac{1}{2}\omega^2 x^2 .$$

Show that the stationary solutions can be written in the form

$$\psi_n(x) = f_n(x)\exp(-\alpha x^2).$$

You should give the value of α and derive any restrictions on $f_n(x)$. Hence determine the possible energy eigenvalues E_n.

The particle has a wave function $\psi(x,t)$ which is even in x at $t = 0$. Write down the general form for $\psi(x,0)$, using the fact that $f_n(x)$ is an even function of x only if n is even. Hence write down $\psi(x,t)$ and show that its probability density is periodic in time with period π/ω.

16C Electromagnetism

A capacitor consists of three perfectly conducting coaxial cylinders of radii a, b and c where $0 < a < b < c$, and length L where $L \gg c$ so that end effects may be ignored. The inner and outer cylinders are maintained at zero potential, while the middle cylinder is held at potential V. Assuming its cylindrical symmetry, compute the electrostatic potential within the capacitor, the charge per unit length on the middle cylinder, the capacitance and the electrostatic energy, both per unit length.

Next assume that the radii a and c are fixed, as is the potential V, while the radius b is allowed to vary. Show that the energy achieves a minimum when b is the geometric mean of a and c.

17B Fluid Dynamics

Starting with the Euler equations for an inviscid incompressible fluid, derive Bernoulli's theorem for unsteady irrotational flow.

Inviscid fluid of density ρ is contained within a U-shaped tube with the arms vertical, of height h and with the same (unit) cross-section. The ends of the tube are closed. In the equilibrium state the pressures in the two arms are p_1 and p_2 and the heights of the fluid columns are ℓ_1 , ℓ_2 .

The fluid in arm 1 is displaced upwards by a distance ξ (and in the other arm downward by the same amount). In the subsequent evolution the pressure above each column may be taken as inversely proportional to the length of tube above the fluid surface. Using Bernoulli's theorem, show that $\xi(t)$ obeys the equation

$$\rho(\ell_1 + \ell_2)\ddot{\xi} + \frac{p_1\xi}{h - \ell_1 - \xi} + \frac{p_2\xi}{h - \ell_2 + \xi} + 2\rho g\xi = 0.$$

Now consider the special case $\ell_1 = \ell_2 = \ell_0$, $p_1 = p_2 = p_0$. Construct a first integral of this equation and hence give an expression for the total kinetic energy $\rho\ell_0\dot{\xi}^2$ of the flow in terms of ξ and the maximum displacement ξ_{max}.

18C Numerical Analysis

Let

$$\langle f, g\rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x)\, g(x)\, dx\,,$$

be an inner product. The Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \ldots$ are polynomials in x of degree n with leading term $2^n x^n$ which are orthogonal with respect to the inner product, with

$$\langle H_m, H_n\rangle = \begin{cases} \gamma_m > 0 & \text{if } m = n\,, \\ 0 & \text{otherwise,} \end{cases}$$

and $H_0(x) = 1$. Find a three-term recurrence relation which is satisfied by $H_n(x)$ and γ_n for $n = 1, 2, 3$. [*You may assume without proof that*

$$\langle 1, 1\rangle = \sqrt{\pi}\,, \quad \langle x, x\rangle = \tfrac{1}{2}\sqrt{\pi}\,, \quad \langle x^2, x^2\rangle = \tfrac{3}{4}\sqrt{\pi}\,, \quad \langle x^3, x^3\rangle = \tfrac{15}{8}\sqrt{\pi}\,.]$$

Next let $x_0, x_1, \ldots, x_k$ be the $k+1$ distinct zeros of $H_{k+1}(x)$ and for $i, j = 0, 1, \ldots, k$ define the Lagrangian polynomials

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

associated with these points. Prove that $\langle L_i, L_j\rangle = 0$ if $i \neq j$.

19E Statistics

Consider the the linear regression model

$$Y_i = \beta\, x_i + \epsilon_i,$$

where the numbers $x_1, \ldots, x_n$ are known, the independent random variables $\epsilon_1, \ldots, \epsilon_n$ have the $N(0, \sigma^2)$ distribution, and the parameters β and σ^2 are unknown. Find the maximum likelihood estimator for β.

State and prove the Gauss–Markov theorem in the context of this model.

Write down the distribution of an arbitrary linear estimator for β. Hence show that there exists a linear, unbiased estimator $\widehat{\beta}$ for β such that

$$\mathbb{E}_{\beta,\sigma^2}[(\widehat{\beta} - \beta)^4] \leqslant \mathbb{E}_{\beta,\sigma^2}[(\widetilde{\beta} - \beta)^4]$$

for all linear, unbiased estimators $\widetilde{\beta}$.

[Hint: If $Z \sim N(a, b^2)$ then $\mathbb{E}\,[(Z - a)^4] = 3\, b^4$.]

20E Markov Chains

Let $(X_n)_{n \geqslant 0}$ be a Markov chain.

(a) What does it mean to say that a state i is positive recurrent? How is this property related to the equilibrium probability π_i? You do not need to give a full proof, but you should carefully state any theorems you use.

(b) What is a communicating class? Prove that if states i and j are in the same communicating class and i is positive recurrent then j is positive recurrent also.

A frog is in a pond with an infinite number of lily pads, numbered $1, 2, 3, \ldots$. She hops from pad to pad in the following manner: if she happens to be on pad i at a given time, she hops to one of pads $(1, 2, \ldots, i, i+1)$ with equal probability.

(c) Find the equilibrium distribution of the corresponding Markov chain.

(d) Now suppose the frog starts on pad k and stops when she returns to it. Show that the expected number of times the frog hops is $e(k-1)!$ where $e = 2.718 \ldots$. What is the expected number of times she will visit the lily pad $k+1$?

END OF PAPER