UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 1 June 2010 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold Cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Write down the matrix representing the following transformations of $\mathbb{R}^3$:

(i) clockwise rotation of $45°$ around the x axis,

(ii) reflection in the plane $x = y$,

(iii) the result of first doing (i) and then (ii).

2D Groups

Express the element $(123)(234)$ in S_5 as a product of disjoint cycles. Show that it is in A_5 . Write down the elements of its conjugacy class in A_5 .

3C Vector Calculus

Consider the vector field

$$\mathbf{F} = (-y/(x^2+y^2),\, x/(x^2+y^2),\, 0)$$

defined on all of $\mathbb{R}^3$ except the z axis. Compute $\boldsymbol{\nabla} \times \mathbf{F}$ on the region where it is defined.

Let γ_1 be the closed curve defined by the circle in the xy-plane with centre $(2, 2, 0)$ and radius 1 , and γ_2 be the closed curve defined by the circle in the xy-plane with centre $(0, 0, 0)$ and radius 1 .

By using your earlier result, or otherwise, evaluate the line integral $\oint_{\gamma_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$.

By explicit computation, evaluate the line integral $\oint_{\gamma_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$. Is your result consistent with Stokes' theorem? Explain your answer briefly.

4C Vector Calculus

A curve in two dimensions is defined by the parameterised Cartesian coordinates

$$x(u) = ae^{bu}\cos u\,, \qquad y(u) = ae^{bu}\sin u\,,$$

where the constants $a, b > 0$. Sketch the curve segment corresponding to the range $0 \leqslant u \leqslant 3\pi$. What is the length of the curve segment between the points $(x(0), y(0))$ and $(x(U), y(U))$, as a function of U?

A geometrically sensitive ant walks along the curve with varying speed $\kappa(u)^{-1}$, where $\kappa(u)$ is the curvature at the point corresponding to parameter u. Find the time taken by the ant to walk from $(\,x(2n\pi),\, y(2n\pi)\,)$ to $(\,x(2(n{+}1)\pi),\, y(2(n{+}1)\pi)\,)$, where n is a positive integer, and hence verify that this time is independent of n.

[*You may quote without proof the formula* $\kappa(u) = \dfrac{|\,x'(u)y''(u) - y'(u)x''(u)\,|}{(\,(x'(u))^2 + (y'(u))^2\,)^{3/2}}$.]

SECTION II

5D Groups

(i) State the orbit-stabilizer theorem.

Let G be the group of rotations of the cube, X the set of faces. Identify the stabilizer of a face, and hence compute the order of G.

Describe the orbits of G on the set $X \times X$ of pairs of faces.

(ii) Define what it means for a subgroup N of G to be *normal*. Show that G has a normal subgroup of order 4.

6D Groups

State Lagrange's theorem. Let p be a prime number. Prove that every group of order p is cyclic. Prove that every abelian group of order p^2 is isomorphic to either $C_p \times C_p$ or C_{p^2}.

Show that D_{12}, the dihedral group of order 12, is not isomorphic to the alternating group A_4.

7D Groups

Let G be a group, X a set on which G acts transitively, B the stabilizer of a point $x \in X$.

Show that if $g \in G$ stabilizes the point $y \in X$, then there exists an $h \in G$ with $hgh^{-1} \in B$.

Let $G = SL_2(\mathbb{C})$, acting on $\mathbb{C} \cup \{\infty\}$ by Möbius transformations. Compute $B = G_\infty$, the stabilizer of ∞. Given

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

compute the set of fixed points $\left\{ x \in \mathbb{C} \cup \{\infty\} \,\middle|\, gx = x \right\}$.

Show that every element of G is conjugate to an element of B.

8D Groups

Let G be a finite group, X the set of proper subgroups of G. Show that conjugation defines an action of G on X.

Let B be a proper subgroup of G. Show that the orbit of G on X containing B has size at most the index $|G : B|$. Show that there exists a $g \in G$ which is not conjugate to an element of B.

9C Vector Calculus

(a) Define a rank two tensor and show that if two rank two tensors A_{ij} and B_{ij} are the same in one Cartesian coordinate system, then they are the same in all Cartesian coordinate systems.

The quantity C_{ij} has the property that, for every rank two tensor A_{ij}, the quantity $C_{ij}A_{ij}$ is a scalar. Is C_{ij} necessarily a rank two tensor? Justify your answer with a proof from first principles, or give a counterexample.

(b) Show that, if a tensor T_{ij} is invariant under rotations about the x_3-axis, then it has the form

$$\begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} .$$

(c) The *inertia tensor* about the origin of a rigid body occupying volume V and with variable mass density $\rho(\mathbf{x})$ is defined to be

$$I_{ij} = \int_V \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j)\,\mathrm{d}V .$$

The rigid body B has uniform density ρ and occupies the cylinder

$$\{ (x_1, x_2, x_3) \, : \, -2 \leqslant x_3 \leqslant 2,\, x_1^2 + x_2^2 \leqslant 1 \} .$$

Show that the inertia tensor of B about the origin is diagonal in the (x_1, x_2, x_3) coordinate system, and calculate its diagonal elements.

10C Vector Calculus

Let $f(x,y)$ be a function of two variables, and R a region in the xy-plane. State the rule for evaluating $\int_R f(x,y)\,dx dy$ as an integral with respect to new variables $u(x,y)$ and $v(x,y)$.

Sketch the region R in the xy-plane defined by

$$R = \{ (x,y) : x^2+y^2 \leqslant 2 ,\ x^2-y^2 \geqslant 1 ,\ x \geqslant 0 ,\ y \geqslant 0 \} .$$

Sketch the corresponding region in the uv-plane, where

$$u = x^2+y^2 , \qquad v = x^2-y^2 .$$

Express the integral

$$I = \int_R (x^5y - xy^5)\exp(4x^2y^2)\,dx\,dy$$

as an integral with respect to u and v. Hence, or otherwise, calculate I.

11C Vector Calculus

State the divergence theorem (also known as Gauss' theorem) relating the surface and volume integrals of appropriate fields.

The surface S_1 is defined by the equation $z = 3 - 2x^2 - 2y^2$ for $1 \leqslant z \leqslant 3$; the surface S_2 is defined by the equation $x^2+y^2 = 1$ for $0 \leqslant z \leqslant 1$; the surface S_3 is defined by the equation $z = 0$ for x, y satisfying $x^2+y^2 \leqslant 1$. The surface S is defined to be the union of the surfaces S_1, S_2 and S_3. Sketch the surfaces S_1, S_2, S_3 and (hence) S.

The vector field $\mathbf{F}$ is defined by

$$\mathbf{F}(x,y,z) = (\, xy + x^6 ,\ -\tfrac{1}{2}y^2 + y^8 ,\ z\,) .$$

Evaluate the integral

$$\oint_S \mathbf{F}\cdot d\mathbf{S} ,$$

where the surface element $d\mathbf{S}$ points in the direction of the outward normal to S.

12C Vector Calculus

Given a spherically symmetric mass distribution with density ρ, explain how to obtain the gravitational field $\mathbf{g} = -\boldsymbol{\nabla}\phi$, where the potential ϕ satisfies Poisson's equation

$$\nabla^2\phi = 4\pi G\rho\,.$$

The remarkable planet Geometria has radius 1 and is composed of an infinite number of stratified spherical shells S_n labelled by integers $n \geqslant 1$. The shell S_n has uniform density $2^{n-1}\rho_0$, where ρ_0 is a constant, and occupies the volume between radius 2^{-n+1} and 2^{-n}.

Obtain a closed form expression for the mass of Geometria.

Obtain a closed form expression for the gravitational field $\mathbf{g}$ due to Geometria at a distance $r = 2^{-N}$ from its centre of mass, for each positive integer $N \geqslant 1$. What is the potential $\phi(r)$ due to Geometria for $r > 1$?

END OF PAPER