UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 28 May 2010 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold Cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Differential Equations

Find the general solutions to the following difference equations for y_n, $n \in \mathbb{N}$.

$$\begin{array}{rrcl} \text{(i)} & y_{n+3} - 3\,y_{n+1} + 2\,y_n &=& 0, \\ \text{(ii)} & y_{n+3} - 3\,y_{n+1} + 2\,y_n &=& 2^n, \\ \text{(iii)} & y_{n+3} - 3\,y_{n+1} + 2\,y_n &=& (-2)^n, \\ \text{(iv)} & y_{n+3} - 3\,y_{n+1} + 2\,y_n &=& (-2)^n + 2^n. \end{array}$$

2A Differential Equations

Let $f(x,y) = g(u,v)$ where the variables $\{x,y\}$ and $\{u,v\}$ are related by a smooth, invertible transformation. State the chain rule expressing the derivatives $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$ in terms of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ and use this to deduce that

$$\frac{\partial^2 g}{\partial u\,\partial v} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\frac{\partial^2 f}{\partial y^2} + H\frac{\partial f}{\partial x} + K\frac{\partial f}{\partial y}$$

where H and K are second-order partial derivatives, to be determined.

Using the transformation $x = uv$ and $y = u/v$ in the above identity, or otherwise, find the general solution of

$$x\,\frac{\partial^2 f}{\partial x^2} - \frac{y^2}{x}\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x} - \frac{y}{x}\frac{\partial f}{\partial y} = 0\,.$$

3F Probability

Jensen's inequality states that for a convex function f and a random variable X with a finite mean, $\mathbb{E}f(X) \geqslant f\big(\mathbb{E}X\big)$.

(a) Suppose that $f(x) = x^m$ where m is a positive integer, and X is a random variable taking values $x_1, \ldots, x_N \geqslant 0$ with equal probabilities, and where the sum $x_1 + \ldots + x_N = 1$. Deduce from Jensen's inequality that

$$\sum_{i=1}^{N} f(x_i) \geqslant N f\left(\frac{1}{N}\right). \qquad (1)$$

(b) N horses take part in m races. The results of different races are independent. The probability for horse i to win any given race is $p_i \geqslant 0$, with $p_1 + \ldots + p_N = 1$.

Let Q be the probability that a single horse wins all m races. Express Q as a polynomial of degree m in the variables $p_1, \ldots, p_N$.

By using (1) or otherwise, prove that $Q \geqslant N^{1-m}$.

4F Probability

Let X and Y be two non-constant random variables with finite variances. The correlation coefficient $\rho(X, Y)$ is defined by

$$\rho(X,Y) = \frac{\mathbb{E}\big[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\big]}{\big(\mathrm{Var}\ X\big)^{1/2}\big(\mathrm{Var}\ Y\big)^{1/2}}.$$

(a) Using the Cauchy–Schwarz inequality or otherwise, prove that

$$-1 \leqslant \rho(X,Y) \leqslant 1.$$

(b) What can be said about the relationship between X and Y when either (i) $\rho(X,Y) = 0$ or (ii) $|\rho(X,Y)| = 1$. [*Proofs are not required.*]

(c) Take $0 \leqslant r \leqslant 1$ and let X, X' be independent random variables taking values ± 1 with probabilities $1/2$. Set

$$Y = \begin{cases} X, & \text{with probability } r, \\ X', & \text{with probability } 1 - r. \end{cases}$$

Find $\rho(X,Y)$.

SECTION II

5A Differential Equations

(a) Consider the differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 , \qquad (1)$$

with $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$. Show that $y(x) = e^{\lambda x}$ is a solution if and only if $p(\lambda) = 0$ where

$$p(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_2 \lambda^2 + a_1 \lambda + a_0 .$$

Show further that $y(x) = x e^{\mu x}$ is also a solution of (1) if μ is a root of the polynomial $p(\lambda)$ of multiplicity at least 2.

(b) By considering $v(t) = \dfrac{d^2 u}{dt^2}$, or otherwise, find the general real solution for $u(t)$ satisfying

$$\frac{d^4 u}{dt^4} + 2 \frac{d^2 u}{dt^2} = 4t^2 . \qquad (2)$$

By using a substitution of the form $u(t) = y(t^2)$ in (2), or otherwise, find the general real solution for $y(x)$, with x positive, where

$$4x^2 \frac{d^4 y}{dx^4} + 12x \frac{d^3 y}{dx^3} + (3 + 2x) \frac{d^2 y}{dx^2} + \frac{dy}{dx} = x .$$

6A Differential Equations

(a) By using a power series of the form

$$y(x) = \sum_{k=0}^{\infty} a_k\, x^k$$

or otherwise, find the general solution of the differential equation

$$xy'' - (1-x)y' - y = 0. \tag{1}$$

(b) Define the Wronskian $W(x)$ for a second order linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

and show that $W' + p(x)W = 0$. Given a non-trivial solution $y_1(x)$ of (2) show that $W(x)$ can be used to find a second solution $y_2(x)$ of (2) and give an expression for $y_2(x)$ in the form of an integral.

(c) Consider the equation (2) with

$$p(x) = -\frac{P(x)}{x} \qquad \text{and} \qquad q(x) = -\frac{Q(x)}{x}$$

where P and Q have Taylor expansions

$$P(x) = P_0 + P_1 x + \dots\,, \qquad Q(x) = Q_0 + Q_1 x + \dots$$

with P_0 a positive integer. Find the roots of the indicial equation for (2) with these assumptions. If $y_1(x) = 1 + \beta x + \dots$ is a solution, use the method of part (b) to find the first two terms in a power series expansion of a linearly independent solution $y_2(x)$, expressing the coefficients in terms of P_0, P_1 and β.

7A Differential Equations

(a) Find the general solution of the system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{1}$$

(b) Depending on the parameter $\lambda \in \mathbb{R}$, find the general solution of the system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2 \begin{pmatrix} -\lambda \\ 1 \\ \lambda \end{pmatrix} e^{2t}, \tag{2}$$

and explain why (2) has a particular solution of the form $\mathbf{c}e^{2t}$ with constant vector $\mathbf{c} \in \mathbb{R}^3$ for $\lambda = 1$ but not for $\lambda \neq 1$.

[*Hint: decompose* $\begin{pmatrix} -\lambda \\ 1 \\ \lambda \end{pmatrix}$ *in terms of the eigenbasis of the matrix in* (1).]

(c) For $\lambda = -1$, find the solution of (2) which goes through the point $(0, 1, 0)$ at $t = 0$.

8A Differential Equations

(a) State how the nature of a critical (or stationary) point of a function $f(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ can be determined by consideration of the eigenvalues of the Hessian matrix H of $f(\mathbf{x})$, assuming H is non-singular.

(b) Let $f(x, y) = xy(1 - x - y)$. Find all the critical points of the function $f(x, y)$ and determine their nature. Determine the zero contour of $f(x, y)$ and sketch a contour plot showing the behaviour of the contours in the neighbourhood of the critical points.

(c) Now let $g(x, y) = x^3y^2(1 - x - y)$. Show that $(0, 1)$ is a critical point of $g(x, y)$ for which the Hessian matrix of g is singular. Find an approximation for $g(x, y)$ to lowest non-trivial order in the neighbourhood of the point $(0, 1)$. Does g have a maximum or a minimum at $(0, 1)$? Justify your answer.

9F Probability

(a) What does it mean to say that a random variable X with values $n = 1, 2, \ldots$ has a geometric distribution with a parameter p where $p \in (0, 1)$?

An expedition is sent to the Himalayas with the objective of catching a pair of wild yaks for breeding. Assume yaks are loners and roam about the Himalayas at random. The probability $p \in (0, 1)$ that a given trapped yak is male is independent of prior outcomes. Let N be the number of yaks that must be caught until a breeding pair is obtained.

(b) Find the expected value of N.

(c) Find the variance of N.

10F Probability

The yearly levels of water in the river Camse are independent random variables X_1, X_2, …, with a given continuous distribution function $F(x) = \mathbb{P}(X_i \leqslant x)$, $x \geqslant 0$ and $F(0) = 0$. The levels have been observed in years 1, …, n and their values X_1, …, X_n recorded. The local council has decided to construct a dam of height

$$Y_n = \max\ \big[X_1, \ldots, X_n\big].$$

Let τ be the subsequent time that elapses before the dam overflows:

$$\tau = \min\ \big[t \geqslant 1 :\ X_{n+t} > Y_n\big].$$

(a) Find the distribution function $\mathbb{P}(Y_n \leqslant z)$, $z > 0$, and show that the mean value $\mathbb{E}Y_n = \displaystyle\int_0^\infty [1 - F(z)^n]\mathrm{d}z$.

(b) Express the conditional probability $\mathbb{P}(\tau = k \,|\, Y_n = z)$, where $k = 1, 2, \ldots$ and $z > 0$, in terms of F.

(c) Show that the unconditional probability

$$\mathbb{P}(\tau = k) = \frac{n}{(k + n - 1)(k + n)}, \quad k = 1, 2, \ldots.$$

(d) Determine the mean value $\mathbb{E}\,\tau$.

11F Probability

In a branching process every individual has probability p_k of producing exactly k offspring, $k = 0, 1, \ldots$, and the individuals of each generation produce offspring independently of each other and of individuals in preceding generations. Let X_n represent the size of the nth generation. Assume that $X_0 = 1$ and $p_0 > 0$ and let $F_n(s)$ be the generating function of X_n. Thus

$$F_1(s) = \mathbb{E}s^{X_1} = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \leqslant 1.$$

(a) Prove that

$$F_{n+1}(s) = F_n(F_1(s)).$$

(b) State a result in terms of $F_1(s)$ about the probability of eventual extinction. [*No proofs are required.*]

(c) Suppose the probability that an individual leaves k descendants in the next generation is $p_k = 1/2^{k+1}$, for $k \geqslant 0$. Show from the result you state in (b) that extinction is certain. Prove further that in this case

$$F_n(s) = \frac{n - (n-1)s}{(n+1) - ns}, \quad n \geqslant 1,$$

and deduce the probability that the nth generation is empty.

12F Probability

Let X_1, X_2 be bivariate normal random variables, with the joint probability density function

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\varphi(x_1,x_2)}{2(1-\rho^2)}\right],$$

where

$$\varphi(x_1,x_2) = \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2$$

and $x_1, x_2 \in \mathbb{R}$.

(a) Deduce that the marginal probability density function

$$f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right].$$

(b) Write down the moment-generating function of X_2 in terms of μ_2 and σ_2. [*No proofs are required.*]

(c) By considering the ratio $f_{X_1,X_2}(x_1,x_2)/f_{X_2}(x_2)$ prove that, conditional on $X_2 = x_2$, the distribution of X_1 is normal, with mean and variance $\mu_1 + \rho\sigma_1(x_2-\mu_2)/\sigma_2$ and $\sigma_1^2(1-\rho^2)$, respectively.

END OF PAPER