17B Methods

Defining the function $G_{f3}(\mathbf{r};\mathbf{r}_0) = -1/(4\pi|\mathbf{r}-\mathbf{r}_0|)$, prove Green's third identity for functions $u(\mathbf{r})$ satisfying Laplace's equation in a volume V with surface S, namely

$$u(\mathbf{r}_0) = \int_S \left(u\frac{\partial G_{f3}}{\partial n} - \frac{\partial u}{\partial n} G_{f3} \right) dS.$$

A solution is sought to the Neumann problem for $\nabla^2 u = 0$ in the half space $z > 0$:

$$u = O(|\mathbf{x}|^{-a}), \quad \frac{\partial u}{\partial r} = O(|\mathbf{x}|^{-a-1}) \text{ as } |\mathbf{x}| \to \infty, \qquad \frac{\partial u}{\partial z} = p(x,y) \text{ on } z = 0,$$

where $a > 1$. It is assumed that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 0$. Explain why this condition is necessary.

Construct an appropriate Green's function $G(\mathbf{r};\mathbf{r}_0)$ satisfying $\partial G/\partial z = 0$ at $z = 0$, using the method of images or otherwise. Hence find the solution in the form

$$u(x_0,y_0,z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) f(x-x_0, y-y_0, z_0)\,dx\,dy,$$

where f is to be determined.

Now let

$$p(x,y) = \begin{cases} x & |x|,|y| < a, \\ 0 & \text{otherwise.} \end{cases}$$

By expanding f in inverse powers of z_0, show that

$$u \to \frac{-2a^4 x_0}{3\pi z_0^3} \qquad \text{as} \quad z_0 \to \infty\,.$$