UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday, 2 June, 2009 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

(1) Let V be a finite-dimensional vector space and let $T : V \to V$ be a non-zero endomorphism of V. If $\ker(T) = \operatorname{im}(T)$ show that the dimension of V is an even integer. Find the minimal polynomial of T. [*You may assume the rank-nullity theorem.*]

(2) Let A_i, $1 \leqslant i \leqslant 3$, be non-zero subspaces of a vector space V with the property that

$$V = A_1 \oplus A_2 = A_2 \oplus A_3 = A_1 \oplus A_3.$$

Show that there is a 2-dimensional subspace $W \subset V$ for which all the $W \cap A_i$ are one-dimensional.

2G Geometry

What is an *ideal hyperbolic triangle*? State a formula for its area.

Compute the area of a hyperbolic disk of hyperbolic radius ρ. Hence, or otherwise, show that no hyperbolic triangle completely contains a hyperbolic circle of hyperbolic radius 2.

3D Complex Analysis or Complex Methods

Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, be an analytic function of z in a domain D of the complex plane. Derive the Cauchy–Riemann equations relating the partial derivatives of u and v.

For $u = e^{-x} \cos y$, find v and hence $f(z)$.

4C Special Relativity

Write down the components of the position four-vector x_μ. Hence find the components of the four-momentum $p_\mu = MU_\mu$ of a particle of mass M, where $U_\mu = dx_\mu/d\tau$, with τ being the proper time.

The particle, viewed in a frame in which it is initially at rest, disintegrates leaving a particle of mass m moving with constant velocity together with other remnants which have a total three-momentum $\mathbf{p}$ and energy E. Show that

$$m = \sqrt{\left(M - \frac{E}{c^2}\right)^2 - \frac{|\mathbf{p}|^2}{c^2}}.$$

5D Fluid Dynamics

A steady velocity field $\mathbf{u} = (u_r, u_\theta, u_z)$ is given in cylindrical polar coordinates (r, θ, z) by

$$u_r = -\alpha r, \quad u_\theta = \frac{\gamma}{r}(1 - e^{-\beta r^2}), \quad u_z = 2\alpha z\,,$$

where α, β, γ are positive constants.

Show that this represents a possible flow of an incompressible fluid, and find the vorticity $\boldsymbol{\omega}$.

Show further that

$$\operatorname{curl}(\mathbf{u} \wedge \boldsymbol{\omega}) = -\nu \nabla^2 \boldsymbol{\omega}$$

for a constant ν which should be calculated.

[*The divergence and curl operators in cylindrical polars are given by*

$$\begin{aligned}
\operatorname{div}\mathbf{u} &= \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \\
\operatorname{curl}\mathbf{u} &= \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \\
\textit{and, when } \boldsymbol{\omega} &= [0, 0, \omega(r,z)], \\
\nabla^2 \boldsymbol{\omega} &= \left[0, 0, \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right) + \frac{\partial^2 \omega}{\partial z^2}\right].
\end{aligned}$$
]

6C Numerical Analysis

The real non-singular matrix $A \in \mathbb{R}^{m\times m}$ is written in the form $A = A_D + A_U + A_L$, where the matrices $A_D, A_U, A_L \in \mathbb{R}^{m\times m}$ are diagonal and non-singular, strictly upper-triangular and strictly lower-triangular respectively.

Given $b \in \mathbb{R}^m$, the Jacobi iteration for solving $Ax = b$ is

$$A_D x_n = -(A_U + A_L)x_{n-1} + b, \qquad n = 1, 2...$$

where the nth iterate is $x_n \in \mathbb{R}^m$. Show that the iteration converges to the solution x of $Ax = b$, independent of the starting choice x_0, if and only if the spectral radius $\rho(H)$ of the matrix $H = -A_D^{-1}(A_U + A_L)$ is less than 1.

Hence find the range of values of the real number μ for which the iteration will converge when

$$A = \begin{bmatrix} 1 & 0 & -\mu \\ -\mu & 3 & -\mu \\ -4\mu & 0 & 4 \end{bmatrix}.$$

7H Statistics

What does it mean to say that an estimator $\hat{\theta}$ of a parameter θ is *unbiased*?

An n-vector Y of observations is believed to be explained by the model

$$Y = X\beta + \varepsilon,$$

where X is a known $n \times p$ matrix, β is an unknown p-vector of parameters, $p < n$, and ε is an n-vector of independent $N(0, \sigma^2)$ random variables. Find the maximum-likelihood estimator $\hat{\beta}$ of β, and show that it is unbiased.

8H Optimization

Find an optimal solution to the linear programming problem

$$\max \; 3x_1 + 2x_2 + 2x_3$$

in $x \geqslant 0$ subject to

$$\begin{aligned} 7x_1 + 3x_2 + 5x_3 &\leqslant 44, \\ x_1 + 2x_2 + x_3 &\leqslant 10, \\ x_1 + x_2 + x_3 &\geqslant 8. \end{aligned}$$

SECTION II

9G Linear Algebra

Define the *dual* of a vector space V. State and prove a formula for its dimension.

Let V be the vector space of real polynomials of degree at most n. If $\{a_0, \ldots, a_n\}$ are distinct real numbers, prove that there are unique real numbers $\{\lambda_0, \ldots, \lambda_n\}$ with

$$\frac{dp}{dx}(0) \;=\; \sum_{j=0}^{n} \lambda_j p(a_j)$$

for every $p(x) \in V$.

10F Groups, Rings and Modules

Prove that a principal ideal domain is a unique factorization domain.

Give, with justification, an example of an element of $\mathbb{Z}[\sqrt{-3}]$ which does not have a unique factorization as a product of irreducibles. Show how $\mathbb{Z}[\sqrt{-3}]$ may be embedded as a subring of index 2 in a ring R (that is, such that the additive quotient group $R/\mathbb{Z}[\sqrt{-3}]$ has order 2) which is a principal ideal domain. [*You should explain why R is a principal ideal domain, but detailed proofs are not required.*]

11E Analysis II

Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \|2^n x\|,$$

where $\|t\|$ is the distance from t to the nearest integer. Prove that f is continuous. [*Results about uniform convergence may not be used unless they are clearly stated and proved.*]

Suppose now that $g : \mathbb{R} \to \mathbb{R}$ is a function which is differentiable at some point x, and let $(u_n)_{n=1}^{\infty}, (v_n)_{n=1}^{\infty}$ be two sequences of real numbers with $u_n \leqslant x \leqslant v_n$ for all n, $u_n \neq v_n$ and $u_n, v_n \to x$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} \frac{g(v_n) - g(u_n)}{v_n - u_n}$$

exists.

By considering appropriate sequences of rationals with denominator 2^{-n}, or otherwise, show that f is nowhere differentiable.

12F Metric and Topological Spaces

Given a function $f : X \to Y$ between metric spaces, we write Γ_f for the subset $\{(x, f(x)) \mid x \in X\}$ of $X \times Y$.

(i) If f is continuous, show that Γ_f is closed in $X \times Y$.

(ii) If Y is compact and Γ_f is closed in $X \times Y$, show that f is continuous.

(iii) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that Γ_f is closed but f is not continuous.

13D Complex Analysis or Complex Methods

Consider the real function $f(t)$ of a real variable t defined by the following contour integral in the complex s-plane:

$$f(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{st}}{(s^2+1)s^{1/2}}\, ds,$$

where the contour Γ is the line $s = \gamma + iy$, $-\infty < y < \infty$, for constant $\gamma > 0$. By closing the contour appropriately, show that

$$f(t) = \sin(t - \pi/4) + \frac{1}{\pi} \int_0^{\infty} \frac{e^{-rt} dr}{(r^2+1)r^{1/2}}$$

when $t > 0$ and is zero when $t < 0$. You should justify your evaluation of the inversion integral over all parts of the contour.

By expanding $(r^2+1)^{-1}\ r^{-1/2}$ as a power series in r, and assuming that you may integrate the series term by term, show that the two leading terms, as $t \to \infty$, are

$$f(t) \sim \sin(t - \pi/4) + \frac{1}{(\pi t)^{1/2}} + \cdots .$$

[*You may assume that* $\int_0^{\infty} x^{-1/2} e^{-x} dx = \pi^{1/2}$.]

14B Mathematical Methods

Find a power series solution about $x = 0$ of the equation

$$xy'' + (1-x)y' + \lambda y = 0,$$

with $y(0) = 1$, and show that y is a polynomial if and only if λ is a non-negative integer n. Let y_n be the solution for $\lambda = n$. Establish an orthogonality relation between y_m and y_n $(m \neq n)$.

Show that $y_m y_n$ is a polynomial of degree $m+n$, and hence that

$$y_m y_n = \sum_{p=0}^{m+n} a_p y_p$$

for appropriate choices of the coefficients a_p and with $a_{m+n} \neq 0$.

For given $n > 0$, show that the functions

$$\{y_m,\ y_m y_n : m = 0, 1, 2, \ldots, n-1\}$$

are linearly independent.

Let $f(x)$ be a polynomial of degree 3. Explain why the expansion

$$f(x) = a_0 y_0(x) + a_1 y_1(x) + a_2 y_2(x) + a_3 y_1(x) y_2(x)$$

holds for appropriate choices of $a_p,\ p = 0, 1, 2, 3$. Hence show that

$$\int_0^\infty e^{-x} f(x)\, dx = w_1 f(\alpha_1) + w_2 f(\alpha_2)\ ,$$

where

$$w_1 = \frac{y_1(\alpha_2)}{y_1(\alpha_2) - y_1(\alpha_1)}\ , \qquad w_2 = \frac{-y_1(\alpha_1)}{y_1(\alpha_2) - y_1(\alpha_1)}\ ,$$

and α_1, α_2 are the zeros of y_2. You need not construct the polynomials $y_1(x), y_2(x)$ explicitly.

15B Quantum Mechanics

A particle of mass m moves in one dimension in a potential $V(x)$ which satisfies $V(x) = V(-x)$. Show that the eigenstates of the Hamiltonian H can be chosen so that they are also eigenstates of the parity operator P. For eigenstates with odd parity $\psi^{odd}(x)$, show that $\psi^{odd}(0) = 0$.

A potential $V(x)$ is given by

$$V(x) = \begin{cases} \kappa\delta(x) & |x| < a \\ \infty & |x| > a \end{cases} .$$

State the boundary conditions satisfied by $\psi(x)$ at $|x| = a$, and show also that

$$\frac{\hbar^2}{2m} \lim_{\epsilon \to 0} \left[\left.\frac{d\psi}{dx}\right|_{\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon} \right] = \kappa\psi(0) .$$

Let the energy eigenstates of even parity be given by

$$\psi^{even}(x) = \begin{cases} A\cos\lambda x \;+\; B\sin\lambda x & -a < x < 0 \\ A\cos\lambda x \;-\; B\sin\lambda x & 0 < x < a \\ 0 & \text{otherwise} \end{cases} .$$

Verify that $\psi^{even}(x)$ satisfies

$$P\psi^{even}(x) = \psi^{even}(x) .$$

By demanding that $\psi^{even}(x)$ satisfy the relevant boundary conditions show that

$$\tan\lambda a = -\frac{\hbar^2}{m}\frac{\lambda}{\kappa} .$$

For $\kappa > 0$ show that the energy eigenvalues E_n^{even}, $n = 0, 1, 2, \ldots$, with $E_n^{even} < E_{n+1}^{even}$, satisfy

$$\eta_n = E_n^{even} - \frac{1}{2m}\left[\frac{(2n+1)\hbar\pi}{2a}\right]^2 > 0.$$

Show also that

$$\lim_{n\to\infty} \eta_n = 0,$$

and give a physical explanation of this result.

Show that the energy eigenstates with odd parity and their energy eigenvalues do not depend on κ.

16A Electromagnetism

Suppose the region $z < 0$ is occupied by an earthed ideal conductor.

(a) Derive the boundary conditions on the tangential electric field $\mathbf{E}$ that hold on the surface $z = 0$.

(b) A point charge q, with mass m, is held above the conductor at $(0, 0, d)$. Show that the boundary conditions on the electric field are satisfied if we remove the conductor and instead place a second charge $-q$ at $(0, 0, -d)$.

(c) The original point charge is now released with zero initial velocity. Ignoring gravity, determine how long it will take for the charge to hit the plane.

17D Fluid Dynamics

A canal has uniform width and a bottom that is horizontal apart from a localised slowly-varying hump of height $D(x)$ whose maximum value is D_{max}. Far upstream the water has depth h_1 and velocity u_1. Show that the depth $h(x)$ of the water satisfies the following equation:

$$\frac{D(x)}{h_1} = 1 - \frac{h}{h_1} - \frac{F}{2}\left(\frac{h_1^2}{h^2} - 1\right) ,$$

where $F = u_1^2/gh_1$.

Describe qualitatively how $h(x)$ varies as the flow passes over the hump in the three cases

(i) $F < 1$ and $D_{max} < D^*$

(ii) $F > 1$ and $D_{max} < D^*$

(iii) $D_{max} = D^*$,

where $D^* = h_1\left(1 - \frac{3}{2}F^{1/3} + \frac{1}{2}F\right)$.

Calculate the water depth far downstream in case (iii) when $F < 1$.

18H Statistics

What is the *critical region* C of a test of the null hypothesis $H_0 : \theta \in \Theta_0$ against the alternative $H_1 : \theta \in \Theta_1$? What is the *size* of a test with critical region C? What is the *power function* of a test with critical region C?

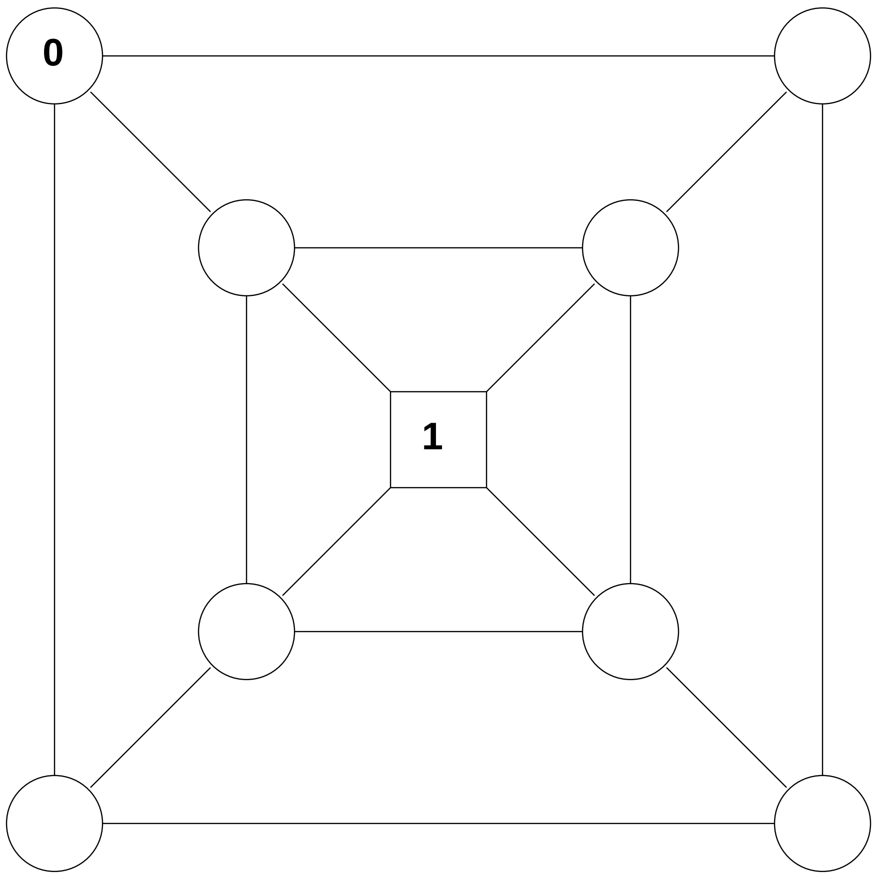
State and prove the Neyman–Pearson Lemma.

If $X_1, \ldots, X_n$ are independent with common $\mathrm{Exp}(\lambda)$ distribution, and $0 < \lambda_0 < \lambda_1$, find the form of the most powerful size-α test of $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda = \lambda_1$. Find the power function as explicitly as you can, and prove that it is increasing in λ. Deduce that the test you have constructed is a size-α test of $H_0 : \lambda \leqslant \lambda_0$ against $H_1 : \lambda = \lambda_1$.

19H Markov Chains

A gerbil is introduced into a maze at the node labelled 0 in the diagram. It roams at random through the maze until it reaches the node labelled 1. At each vertex, it chooses to move to one of the neighbouring nodes with equal probability, independently of all other choices. Find the mean number of moves required for the gerbil to reach node 1.

Suppose now that the gerbil is intelligent, in that when it reaches a node it will not immediately return to the node from which it has just come, choosing with equal probability from all other neighbouring nodes. Express the movement of the gerbil in terms of a Markov chain whose states and transition probabilities you should specify. Find the mean number of moves until the intelligent gerbil reaches node 1. Compare with your answer to the first part, and comment briefly.



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