UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 1 June, 2009 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A*** *and* ***E*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

Let R_1 and R_2 be relations on a set A. Let us say that R_2 *extends* R_1 if xR_1y implies that xR_2y. If R_2 extends R_1, then let us call R_2 an *extension* of R_1.

Let Q be a relation on a set A. Let R be the extension of Q defined by taking xRy if and only if xQy or $x = y$. Let S be the extension of R defined by taking xSy if and only if xRy or yRx. Finally, let T be the extension of S defined by taking xTy if and only if there is a positive integer n and a sequence $(x_0, x_1, \ldots, x_n)$ such that $x_0 = x$, $x_n = y$, and $x_{i-1}Sx_i$ for each i from 1 to n.

Prove that R is reflexive, S is reflexive and symmetric, and T is an equivalence relation.

Let E be any equivalence relation that extends Q. Prove that E extends T.

2E Numbers and Sets

(a) Find integers x and y such that

$$9x + 12y \equiv 4 \pmod{47} \qquad \text{and} \qquad 6x + 7y \equiv 14 \pmod{47}.$$

(b) Calculate $43^{135} \pmod{137}$.

3A Dynamics and Relativity

A rocket moves vertically upwards in a uniform gravitational field and emits exhaust gas downwards with time-dependent speed $U(t)$ relative to the rocket. Derive the rocket equation

$$m(t)\frac{\mathrm{d}v}{\mathrm{d}t} + U(t)\frac{\mathrm{d}m}{\mathrm{d}t} = -m(t)g\,,$$

where $m(t)$ and $v(t)$ are respectively the rocket's mass and upward vertical speed at time t. Suppose now that $m(t) = m_0 - \alpha t$, $U(t) = U_0 m_0/m(t)$ and $v(0) = 0$. What is the condition for the rocket to lift off at $t = 0$? Assuming that this condition is satisfied, find $v(t)$.

State the dimensions of all the quantities involved in your expression for $v(t)$, and verify that the expression is dimensionally consistent.

[*You may assume that all speeds are small compared with the speed of light and neglect any relativistic effects.*]

4A Dynamics and Relativity

(a) Explain what is meant by a *central force* acting on a particle moving in three dimensions.

(b) Show that the orbit of a particle experiencing a central force lies in a plane.

(c) Show that, in the approximation in which the Sun is regarded as fixed and only its gravitational field is considered, a straight line joining the Sun and an orbiting planet sweeps out equal areas in equal times (Kepler's second law).

[*With respect to the basis vectors* $(\mathbf{e}_r, \mathbf{e}_\theta)$ *of plane polar coordinates, the velocity* $\dot{\mathbf{x}}$ *and acceleration* $\ddot{\mathbf{x}}$ *of a particle are given by* $\dot{\mathbf{x}} = (\dot{r}, r\dot{\theta})$ *and* $\ddot{\mathbf{x}} = (\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\dot{\theta})$.]

SECTION II

5E Numbers and Sets

(a) Let A and B be non-empty sets and let $f : A \to B$.

Prove that f is an injection if and only if f has a left inverse.

Prove that f is a surjection if and only if f has a right inverse.

(b) Let A, B and C be sets and let $f : B \to A$ and $g : B \to C$ be functions. Suppose that f is a surjection. Prove that there is a function $h : A \to C$ such that for every $a \in A$ there exists $b \in B$ with $f(b) = a$ and $g(b) = h(a)$.

Prove that h is unique if and only if $g(b) = g(b')$ whenever $f(b) = f(b')$.

6E Numbers and Sets

(a) State and prove the inclusion–exclusion formula.

(b) Let k and m be positive integers, let $n = km$, let $A_1, \ldots, A_k$ be disjoint sets of size m, and let $A = A_1 \cup \ldots \cup A_k$. Let $\mathcal{B}$ be the collection of all subsets $B \subset A$ with the following two properties:

(i) $|B| = k$;

(ii) there is at least one i such that $|B \cap A_i| = 3$.

Prove that the number of sets in $\mathcal{B}$ is given by the formula

$$\sum_{r=1}^{\lfloor k/3 \rfloor} (-1)^{r-1} \binom{k}{r} \binom{m}{3}^r \binom{n-rm}{k-3r}.$$

7E Numbers and Sets

Let p be a prime number and let $\mathbb{Z}_p$ denote the set of integers modulo p. Let k be an integer with $0 \leqslant k \leqslant p$ and let A be a subset of $\mathbb{Z}_p$ of size k.

Let t be a non-zero element of $\mathbb{Z}_p$. Show that if $a + t \in A$ whenever $a \in A$ then $k = 0$ or $k = p$. Deduce that if $1 \leqslant k \leqslant p-1$, then the sets $A, A+1, \ldots, A+p-1$ are all distinct, where $A + t$ denotes the set $\{a + t : a \in A\}$. Deduce from this that $\binom{p}{k}$ is a multiple of p whenever $1 \leqslant k \leqslant p-1$.

Now prove that $(a+1)^p = a^p + 1$ for any $a \in \mathbb{Z}_p$, and use this to prove Fermat's little theorem. Prove further that if $Q(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial in x with coefficients in $\mathbb{Z}_p$, then the polynomial $(Q(x))^p$ is equal to $a_n x^{pn} + a_{n-1} x^{p(n-1)} + \ldots + a_1 x^p + a_0$.

8E Numbers and Sets

Prove that the set of all infinite sequences $(\epsilon_1, \epsilon_2, \ldots)$ with every ϵ_i equal to 0 or 1 is uncountable. Deduce that the closed interval $[0,1]$ is uncountable.

For an ordered set X let $\Sigma(X)$ denote the set of increasing (but not necessarily strictly increasing) sequences in X that are bounded above. For each of $\Sigma(\mathbb{Z})$, $\Sigma(\mathbb{Q})$ and $\Sigma(\mathbb{R})$, determine (with proof) whether it is uncountable.

9A Dynamics and Relativity

Davros departs on a rocket voyage from the planet Skaro, travelling at speed u (where $0 < u < c$) in the positive x direction in Skaro's rest frame. After travelling a distance L in Skaro's rest frame, he jumps onto another rocket travelling at speed v' (where $0 < v' < c$) in the positive x direction in the first rocket's rest frame. After travelling a further distance L in Skaro's rest frame, he jumps onto a third rocket, travelling at speed w'' (where $0 < w'' < c$) in the negative x direction in the second rocket's rest frame.

Let v and w be Davros' speed on the second and third rockets, respectively, in Skaro's rest frame. Show that

$$v = (u + v')\left(1 + \frac{uv'}{c^2}\right)^{-1} .$$

Express w in terms of u, v', w'' and c.

How large must w'' be, expressed in terms of u, v' and c, to ensure that Davros eventually returns to Skaro?

Supposing that w'' satisfies this condition, draw a spacetime diagram illustrating Davros' journey. Label clearly each point where he boards a rocket and the point of his return to Skaro, and give the coordinates of each point in Skaro's rest frame, expressed in terms of u, v, w, c and L.

Hence, or otherwise, calculate how much older Davros will be on his return, and how much time will have elapsed on Skaro during his voyage, giving your answers in terms of u, v, w, c and L.

[*You may neglect any effects due to gravity and any corrections arising from Davros' brief accelerations when getting onto or leaving rockets.*]

10A Dynamics and Relativity

(a) Write down expressions for the relativistic 3-momentum $\mathbf{p}$ and energy E of a particle of rest mass m and velocity $\mathbf{v}$. Show that these expressions are consistent with

$$E^2 = \mathbf{p}.\mathbf{p}\, c^2 + m^2 c^4 \,. \qquad (*)$$

Define the 4-momentum $\mathbf{P}$ for such a particle and obtain $(*)$ by considering the invariance properties of $\mathbf{P}$.

(b) Two particles, each with rest mass m and energy E, moving in opposite directions, collide head on. Show that it is consistent with the conservation of 4-momentum for the collision to result in a set of n particles of rest masses μ_i (for $1 \leqslant i \leqslant n$) only if

$$E \geqslant \frac{1}{2} \left(\sum_{i=1}^{n} \mu_i \right) c^2 \,.$$

(c) A particle of rest mass m_1 and energy E_1 is fired at a stationary particle of rest mass m_2. Show that it is consistent with the conservation of 4-momentum for the collision to result in a set of n particles of rest masses μ_i (for $1 \leqslant i \leqslant n$) only if

$$E_1 \geqslant \frac{(\sum_{i=1}^{n} \mu_i)^2 - m_1^2 - m_2^2}{2m_2} \, c^2 \,.$$

Deduce the minimum frequency required for a photon fired at a stationary particle of rest mass m_2 to result in the same set of n particles, assuming that the conservation of 4-momentum is the only relevant constraint.

11A Dynamics and Relativity

Obtain the moment of inertia of a uniform disc of radius a and mass M about its axis of rotational symmetry. A uniform rigid body of mass $3M/4$ takes the form of a disc of radius a with a concentric circular hole of radius $a/2$ cut out. Calculate the body's moment of inertia about its axis of rotational symmetry.

The body rolls without slipping, with its axis of symmetry horizontal, down a plane inclined at angle α to the horizontal. Determine its acceleration and the frictional and normal-reaction forces resulting from contact with the plane.

12A Dynamics and Relativity

(a) A particle of charge q moves with velocity $\mathbf{v}$ in a constant magnetic field $\mathbf{B}$. Give an expression for the Lorentz force $\mathbf{F}$ experienced by the particle. If no other forces act on the particle, show that its kinetic energy is independent of time.

(b) Four point particles, each of positive charge Q, are fixed at the four corners of a square with sides of length $2a$. Another point particle, of positive charge q, is constrained to move in the plane of the square but is otherwise free.

By considering the form of the electrostatic potential near the centre of the square, show that the state in which the particle of charge q is stationary at the centre of the square is a stable equilibrium. Obtain the frequency of small oscillations about this equilibrium.

[*The Coulomb potential for two point particles of charges* Q *and* q *separated by distance* r *is* $Qq/4\pi\epsilon_0 r$.]

END OF PAPER