UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday, 29 May, 2009 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***C*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Differential Equations

The size of the population of ducks living on the pond of a certain Cambridge college is governed by the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \alpha N - N^2,$$

where $N = N(t)$ is the number of ducks at time t and α is a positive constant. Given that $N(0) = 2\alpha$, find $N(t)$. What happens as $t \to \infty$?

2C Differential Equations

Solve the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \mathrm{e}^{3x}$$

subject to the conditions $y = \mathrm{d}y/\mathrm{d}x = 0$ when $x = 0$.

3F Probability

Consider a pair of jointly normal random variables X_1, X_2, with mean values μ_1, μ_2, variances σ_1^2, σ_2^2 and correlation coefficient ρ with $|\rho| < 1$.

(a) Write down the joint probability density function for (X_1, X_2).

(b) Prove that X_1, X_2 are independent if and only if $\rho = 0$.

4F Probability

Prove the law of total probability: if A_1, …, A_n are pairwise disjoint events with $\mathbb{P}(A_i) > 0$, and $B \subseteq A_1 \cup \ldots \cup A_n$ then $\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B|A_i)$.

There are n people in a lecture room. Their birthdays are independent random variables, and each person's birthday is equally likely to be any of the 365 days of the year. By using the bound $1 - x \leqslant e^{-x}$ for $0 \leqslant x \leqslant 1$, prove that if $n \geqslant 29$ then the probability that at least two people have the same birthday is at least $2/3$.

[*In calculations, you may take* $\sqrt{1 + 8 \times 365 \ln 3} = 56.6$.]

SECTION II

5C Differential Equations

Consider the first-order ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f_1(x)y + f_2(x)y^p, \qquad (*)$$

where $y \geqslant 0$ and p is a positive constant with $p \neq 1$. Let $u = y^{1-p}$. Show that u satisfies

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (1-p)[f_1(x)u + f_2(x)].$$

Hence, find the general solution of equation $(*)$ when $f_1(x) = 1,\ f_2(x) = x$.

Now consider the case $f_1(x) = 1,\ f_2(x) = -\alpha^2$, where α is a non-zero constant. For $p > 1$ find the two equilibrium points of equation $(*)$, and determine their stability. What happens when $0 < p < 1$?

6C Differential Equations

Consider the second-order ordinary differential equation

$$\ddot{x} + 2k\dot{x} + \omega^2 x = 0\ ,$$

where $x = x(t)$ and k and ω are constants with $k > 0$. Calculate the general solution in the cases (i) $k < \omega$, (ii) $k = \omega$, (iii) $k > \omega$.

Now consider the system

$$\ddot{x} + 2k\dot{x} + \omega^2 x = \begin{cases} a & \text{when } \dot{x} > 0 \\ 0 & \text{when } \dot{x} \leqslant 0 \end{cases}$$

with $x(0) = x_1$, $\dot{x}(0) = 0$, where a and x_1 are positive constants. In the case $k < \omega$ find $x(t)$ in the ranges $0 \leqslant t \leqslant \pi/p$ and $\pi/p \leqslant t \leqslant 2\pi/p$, where $p = \sqrt{\omega^2 - k^2}$. Hence, determine the value of x_1 for which $x(t)$ is periodic. For $k > \omega$ can $x(t)$ ever be periodic? Justify your answer.

7C Differential Equations

Consider the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(c-x)\frac{\mathrm{d}y}{\mathrm{d}x}-y=0\,,$$

where c is a constant with $0<c<1$. Determine two linearly independent series solutions about $x=0$, giving an explicit expression for the coefficient of the general term in each series.

Determine the solution of

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(c-x)\frac{\mathrm{d}y}{\mathrm{d}x}-y=x$$

for which $y(0)=0$ and $\mathrm{d}y/\mathrm{d}x$ is finite at $x=0$.

8C Differential Equations

(a) The function $y(x,t)$ satisfies the forced wave equation

$$\frac{\partial^2 y}{\partial x^2}-\frac{\partial^2 y}{\partial t^2}=4$$

with initial conditions $y(x,0)=\sin x$ and $\partial y/\partial t(x,0)=0$. By making the change of variables $u=x+t$ and $v=x-t$, show that

$$\frac{\partial^2 y}{\partial u\partial v}=1\,.$$

Hence, find $y(x,t)$.

(b) The thickness of an axisymmetric drop of liquid spreading on a flat surface satisfies

$$\frac{\partial h}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right),$$

where $h=h(r,t)$ is the thickness of the drop, r is the radial coordinate on the surface and t is time. The drop has radius $R(t)$. The boundary conditions are that $\partial h/\partial r=0$ at $r=0$ and $h(r,t)\propto(R(t)-r)^{1/3}$ as $r\to R(t)$.

Show that

$$M=\int_0^{R(t)} rh\,\mathrm{d}r$$

is independent of time. Given that $h(r,t)=f(r/t^{\alpha})t^{-1/4}$ for some function f (which need not be determined) and that $R(t)$ is proportional to t^{α}, find α.

9F Probability

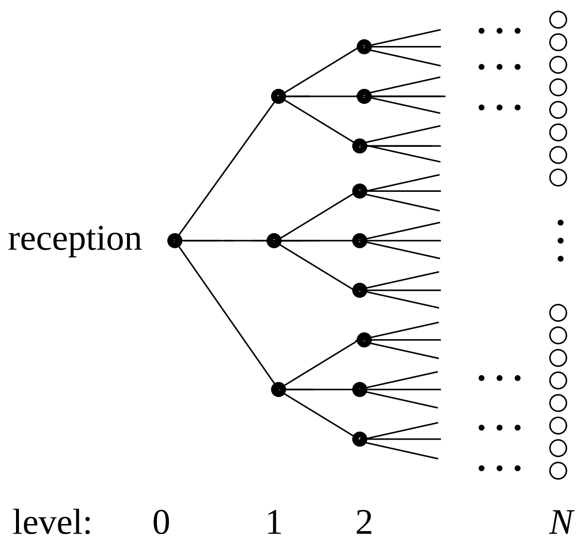
I throw two dice and record the scores S_1 and S_2. Let X be the sum $S_1 + S_2$ and Y the difference $S_1 - S_2$.

(a) Suppose that the dice are fair, so the values $1, \ldots, 6$ are equally likely. Calculate the mean and variance of both X and Y. Find all the values of x and y at which the probabilities $\mathbb{P}(X = x)$, $\mathbb{P}(Y = y)$ are each either greatest or least. Determine whether the random variables X and Y are independent.

(b) Now suppose that the dice are unfair, and that they give the values $1, \ldots, 6$ with probabilities $p_1, \ldots, p_6$ and $q_1, \ldots, q_6$, respectively. Write down the values of $\mathbb{P}(X = 2)$, $\mathbb{P}(X = 7)$ and $\mathbb{P}(X = 12)$. By comparing $\mathbb{P}(X = 7)$ with $\sqrt{\mathbb{P}(X = 2)\,\mathbb{P}(X = 12)}$ and applying the arithmetic-mean–geometric-mean inequality, or otherwise, show that the probabilities $\mathbb{P}(X = 2)$, $\mathbb{P}(X = 3)$, …, $\mathbb{P}(X = 12)$ cannot all be equal.

10F Probability

No-one in their right mind would wish to be a guest at the Virtual Reality Hotel. See the diagram below showing a part of the floor plan of the hotel where rooms are represented by black or white circles. The hotel is built in a shape of a tree: there is one room (reception) situated at level 0, three rooms at level 1, nine at level 2, and so on. The rooms are joined by corridors to their neighbours: each room has four neighbours, apart from the reception, which has three neighbours. Each corridor is blocked with probability $1/3$ and open for passage in both directions with probability $2/3$, independently for different corridors. Every room at level N, where N is a given very large number, has an open window through which a guest can (and should) escape into the street. An arriving guest is placed in the reception and then wanders freely, insofar as the blocked corridors allow.



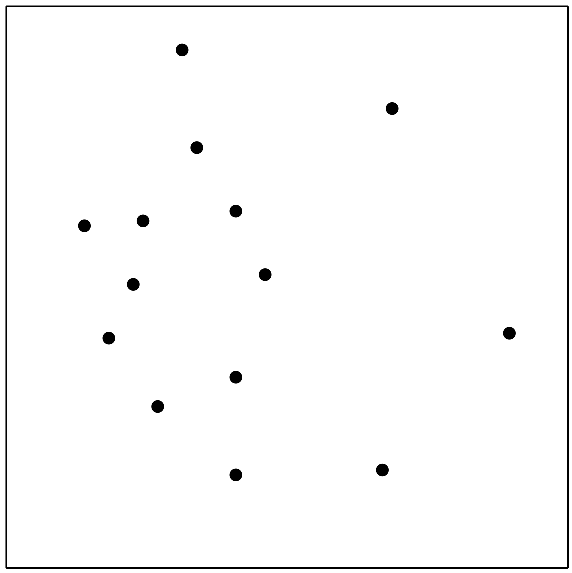
(a) Prove that the probability that the guest will not escape is close to a solution of the equation $\phi(t) = t$, where $\phi(t)$ is a probability-generating function that you should specify.

(b) Hence show that the guest's chance of escape is approximately $(9 - 3\sqrt{3})/4$.

11F Probability

Let X and Y be two independent uniformly distributed random variables on $[0,1]$. Prove that $\mathbb{E}X^k = \dfrac{1}{k+1}$ and $\mathbb{E}(XY)^k = \dfrac{1}{(k+1)^2}$, and find $\mathbb{E}(1-XY)^k$, where k is a non-negative integer.

Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be n independent random points of the unit square $\mathcal{S} = \{(x,y):\ 0 \leqslant x,y \leqslant 1\}$. We say that (X_i,Y_i) is a *maximal external point* if, for each $j=1,\ldots,n$, either $X_j \leqslant X_i$ or $Y_j \leqslant Y_i$. (For example, in the figure below there are three maximal external points.) Determine the expected number of maximal external points.



12F Probability

Let A_1, A_2 and A_3 be three pairwise disjoint events such that the union $A_1 \cup A_2 \cup A_3$ is the full event and $\mathbb{P}(A_1), \mathbb{P}(A_2), \mathbb{P}(A_3) > 0$. Let E be any event with $\mathbb{P}(E) > 0$. Prove the formula

$$\mathbb{P}(A_i|E) = \frac{\mathbb{P}(A_i)\mathbb{P}(E|A_i)}{\sum\limits_{j=1,2,3} \mathbb{P}(A_j)\mathbb{P}(E|A_j)}\,.$$

A Royal Navy speedboat has intercepted an abandoned cargo of packets of the deadly narcotic spitamin. This sophisticated chemical can be manufactured in only three places in the world: a plant in Authoristan (A), a factory in Bolimbia (B) and the ultramodern laboratory on board of a pirate submarine Crash (C) cruising ocean waters. The investigators wish to determine where this particular cargo comes from, but in the absence of prior knowledge they have to assume that each of the possibilities A, B and C is equally likely.

It is known that a packet from A contains pure spitamin in 95% of cases and is contaminated in 5% of cases. For B the corresponding figures are 97% and 3%, and for C they are 99% and 1%.

Analysis of the captured cargo showed that out of 10000 packets checked, 9800 contained the pure drug and the remaining 200 were contaminated. On the basis of this analysis, the Royal Navy captain estimated that 98% of the packets contain pure spitamin and reported his opinion that with probability roughly 0.5 the cargo was produced in B and with probability roughly 0.5 it was produced in C.

Assume that the number of contaminated packets follows the binomial distribution $\mathrm{Bin}(10000, \delta/100)$ where δ equals 5 for A, 3 for B and 1 for C. Prove that the captain's opinion is wrong: there is an overwhelming chance that the cargo comes from B.

[**Hint:** *Let* E *be the event that* 200 *out of* 10000 *packets are contaminated. Compare the ratios of the conditional probabilities* $\mathbb{P}(E|A)$, $\mathbb{P}(E|B)$ *and* $\mathbb{P}(E|C)$. *You may find it helpful that* $\ln 3 \approx 1.09861$ *and* $\ln 5 \approx 1.60944$. *You may also take* $\ln(1-\delta/100) \approx -\delta/100$.]

END OF PAPER