UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 28 May, 2009 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

*The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt **all four** questions from Section I and **at most five** questions from Section II. In Section II, no more than **three** questions on each course may be attempted.*

Complete answers are preferred to fragments.

*Write on **one** side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles, marked **A**, **B**, **C**, **D**, **E** and **F** according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold Cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Vectors and Matrices

Describe geometrically the three sets of points defined by the following equations in the complex z plane:

(a) $z\overline{\alpha} + \overline{z}\alpha = 0$, where α is non-zero;

(b) $2|z - a| = z + \overline{z} + 2a$, where a is real and non-zero;

(c) $\log z = \mathrm{i} \log \overline{z}$.

2B Vectors and Matrices

Define the Hermitian conjugate $A^\dagger$ of an $n \times n$ complex matrix A. State the conditions (i) for A to be Hermitian (ii) for A to be unitary.

In the following, A, B, C and D are $n \times n$ complex matrices and $\mathbf{x}$ is a complex n-vector. A matrix N is defined to be *normal* if $N^\dagger N = N N^\dagger$.

(a) Let A be nonsingular. Show that $B = A^{-1}A^\dagger$ is unitary if and only if A is normal.

(b) Let C be normal. Show that $|C\mathbf{x}| = 0$ if and only if $|C^\dagger \mathbf{x}| = 0$.

(c) Let D be normal. Deduce from (b) that if $\mathbf{e}$ is an eigenvector of D with eigenvalue λ then $\mathbf{e}$ is also an eigenvector of $D^\dagger$ and find the corresponding eigenvalue.

3F Analysis I

Determine the limits as $n \to \infty$ of the following sequences:

(a) $a_n = n - \sqrt{n^2 - n}\,$;

(b) $b_n = \cos^2\left(\pi\sqrt{n^2 + n}\right)$.

4E Analysis I

Let $a_0, a_1, a_2, \ldots$ be a sequence of complex numbers. Prove that there exists $R \in [0, \infty]$ such that the power series $\sum_{n=0}^{\infty} a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$.

Give an example of a power series $\sum_{n=0}^{\infty} a_n z^n$ that diverges if $z = \pm 1$ and converges if $z = \pm \mathrm{i}$.

SECTION II

5C Vectors and Matrices

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unit vectors. By using suffix notation, prove that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c}) = \mathbf{b}\cdot\mathbf{c} - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c}) \qquad (1)$$

and

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c}) = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{a}\ . \qquad (2)$$

The three distinct points A, B, C with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ lie on the surface of the unit sphere centred on the origin O. The *spherical distance* between the points A and B, denoted $\delta(A,B)$, is the length of the (shorter) arc of the circle with centre O passing through A and B. Show that

$$\cos\delta(A,B) = \mathbf{a}\cdot\mathbf{b}\ .$$

A *spherical triangle* with vertices A, B, C is a region on the sphere bounded by the three circular arcs AB, BC, CA. The interior angles of a spherical triangle at the vertices A, B, C are denoted α, β, γ, respectively.

By considering the normals to the planes OAB and OAC, or otherwise, show that

$$\cos\alpha = \frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})}{|\mathbf{a}\times\mathbf{b}||\mathbf{a}\times\mathbf{c}|}.$$

Using identities (1) and (2), prove that

$$\cos\delta(B,C) = \cos\delta(A,B)\cos\delta(A,C) + \sin\delta(A,B)\sin\delta(A,C)\cos\alpha$$

and

$$\frac{\sin\alpha}{\sin\delta(B,C)} = \frac{\sin\beta}{\sin\delta(A,C)} = \frac{\sin\gamma}{\sin\delta(A,B)}.$$

For an equilateral spherical triangle show that $\alpha > \pi/3$.

6B Vectors and Matrices

Explain why the number of solutions $\mathbf{x} \in \mathbb{R}^3$ of the matrix equation $A\mathbf{x} = \mathbf{c}$ is 0, 1 or infinity, where A is a real 3×3 matrix and $\mathbf{c} \in \mathbb{R}^3$. State conditions on A and $\mathbf{c}$ that distinguish between these possibilities, and state the relationship that holds between any two solutions when there are infinitely many.

Consider the case

$$A = \begin{pmatrix} a & a & b \\ b & a & a \\ a & b & a \end{pmatrix} \quad \text{and } \mathbf{c} = \begin{pmatrix} 1 \\ c \\ 1 \end{pmatrix} .$$

Use row and column operations to find and factorize the determinant of A.

Find the kernel and image of the linear map represented by A for all values of a and b. Find the general solution to $A\mathbf{x} = \mathbf{c}$ for all values of a, b and c for which a solution exists.

7A Vectors and Matrices

Let A be an $n \times n$ Hermitian matrix. Show that all the eigenvalues of A are real.

Suppose now that A has n distinct eigenvalues.

(a) Show that the eigenvectors of A are orthogonal.

(b) Define the *characteristic polynomial* $P_A(t)$ of A. Let

$$P_A(t) = \sum_{r=0}^{n} a_r t^r \,.$$

Prove the matrix identity

$$\sum_{r=0}^{n} a_r A^r = 0 \,.$$

(c) What is the range of possible values of

$$\frac{\mathbf{x}^\dagger A \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}}$$

for non-zero vectors $\mathbf{x} \in \mathbb{C}^n$? Justify your answer.

(d) For any (not necessarily symmetric) real 2×2 matrix B with real eigenvalues, let $\lambda_{\max}(B)$ denote its maximum eigenvalue. Is it possible to find a constant C such that

$$\frac{\mathbf{x}^\dagger B \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}} \leqslant C\, \lambda_{\max}(B)$$

for all non-zero vectors $\mathbf{x} \in \mathbb{R}^2$ and all such matrices B? Justify your answer.

8A Vectors and Matrices

(a) Explain what is meant by saying that a 2×2 real transformation matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserves the scalar product with respect to the Euclidean metric $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{R}^2$.

Derive a description of all such matrices that uses a single real parameter together with choices of sign (± 1). Show that these matrices form a group.

(b) Explain what is meant by saying that a 2×2 real transformation matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserves the scalar product with respect to the Minkowski metric $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{R}^2$.

Consider now the set of such matrices with $a > 0$. Derive a description of all matrices in this set that uses a single real parameter together with choices of sign (± 1). Show that these matrices form a group.

(c) What is the intersection of these two groups?

9F Analysis I

For each of the following series, determine for which real numbers x it diverges, for which it converges, and for which it converges absolutely. Justify your answers briefly.

(a) $$\sum_{n \geqslant 1} \frac{3 + \left(\sin x\right)^n}{n} \left(\sin x\right)^n,$$

(b) $$\sum_{n \geqslant 1} \left|\sin x\right|^n \frac{(-1)^n}{\sqrt{n}},$$

(c) $$\sum_{n \geqslant 1} \underbrace{\sin\left(0.99 \ \sin\left(0.99 \ldots \sin\left(0.99\ x\right)\ldots\right)\right)}_{n \text{ times}}.$$

10D Analysis I

State and prove the intermediate value theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $P = (a, b)$ be a point of the plane $\mathbb{R}^2$. Show that the set of distances from points $(x, f(x))$ on the graph of f to the point P is an interval $[A, \infty)$ for some value $A \geqslant 0$.

11D Analysis I

State and prove Rolle's theorem.

Let f and g be two continuous, real-valued functions on a closed, bounded interval $[a,b]$ that are differentiable on the open interval (a,b). By considering the determinant

$$\phi(x) = \begin{vmatrix} 1 & 1 & 0 \\ f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \end{vmatrix} = g(x)\,(f(b)-f(a)) - f(x)\,(g(b)-g(a)) \ ,$$

or otherwise, show that there is a point $c \in (a,b)$ with

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a)) \ .$$

Suppose that $f, g : (0,\infty) \to \mathbb{R}$ are differentiable functions with $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$. Prove carefully that if the limit $\lim_{x\to 0} \dfrac{f'(x)}{g'(x)} = \ell$ exists and is finite, then the limit $\lim_{x\to 0} \dfrac{f(x)}{g(x)}$ also exists and equals ℓ.

12E Analysis I

(a) What does it mean for a function $f : [a,b] \to \mathbb{R}$ to be *Riemann integrable*?

(b) Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Suppose that for every $\delta > 0$ there is a sequence

$$0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \ldots \leqslant a_n < b_n \leqslant 1$$

such that for each i the function f is Riemann integrable on the closed interval $[a_i, b_i]$, and such that $\sum_{i=1}^{n}(b_i - a_i) \geqslant 1 - \delta$. Prove that f is Riemann integrable on $[0,1]$.

(c) Let $f : [0,1] \to \mathbb{R}$ be defined as follows. We set $f(x) = 1$ if x has an infinite decimal expansion that consists of 2s and 7s only, and otherwise we set $f(x) = 0$. Prove that f is Riemann integrable and determine $\int_0^1 f(x)\,\mathrm{d}x$.

END OF PAPER