UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday 6 June 2008 9.00 to 12.00

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Let p be an odd prime number. Assuming that the multiplicative group of $\mathbb{Z}/p\mathbb{Z}$ is cyclic, prove that the multiplicative group of units of $\mathbb{Z}/p^n\mathbb{Z}$ is cyclic for all $n \geqslant 1$.

Find an integer a such that its residue class in $\mathbb{Z}/11^n\mathbb{Z}$ generates the multiplicative group of units for all $n \geqslant 1$.

2F Topics in Analysis

(a) State Runge's theorem on uniform approximation of analytic functions by polynomials.

(b) Suppose f is analytic on

$$\Omega = \{z \in \mathbf{C} : |z| < 1\} \setminus \{z \in \mathbf{C} : \mathrm{Im}(z) = 0,\ \mathrm{Re}(z) \leqslant 0\}.$$

Prove that there exists a sequence of polynomials which converges to f uniformly on compact subsets of Ω.

3G Geometry of Group Actions

Define the hyperbolic metric (in the sense of metric spaces) on the 3-ball.

Given a finite set in hyperbolic 3-space, show there is at least one closed ball of minimal radius containing that set.

4G Coding and Cryptography

What is a binary *cyclic* code of length N? What is the generator polynomial for such a cyclic code? Prove that the generator polynomial is a factor of $X^N - 1$ over the field $\mathbb{F}_2$.

Find all the binary cyclic codes of length 5 .

5J Statistical Modelling

A long-term agricultural experiment had $n = 90$ grassland plots, each 25m $\times$ 25m, differing in biomass, soil pH, and species richness (the count of species in the whole plot). While it was well-known that species richness declines with increasing biomass, it was not known how this relationship depends on soil pH. In the experiment, there were 30 plots of "low pH", 30 of "medium pH" and 30 of "high pH". Three lines of the data are reproduced here as an aid.

```
> grass[c(1,31, 61), ]
      pH   Biomass Species
1   high 0.4692972      30
31   mid 0.1757627      29
61   low 0.1008479      18
```

Briefly explain the commands below. That is, explain the models being fitted.

```
> fit1 <- glm(Species ~ Biomass, family = poisson)
> fit2 <- glm(Species ~ pH + Biomass, family = poisson)
> fit3 <- glm(Species ~ pH * Biomass, family = poisson)
```

Let H_1, H_2 and H_3 denote the hypotheses represented by the three models and fits. Based on the output of the code below, what hypotheses are being tested, and which of the models seems to give the best fit to the data? Why?

```
> anova(fit1, fit2, fit3, test = "Chisq")
Analysis of Deviance Table
Model 1: Species ~ Biomass
Model 2: Species ~ pH + Biomass
Model 3: Species ~ pH * Biomass
  Resid. Df Resid. Dev Df Deviance P(>|Chi|)
1        88     407.67
2        86      99.24  2   308.43 1.059e-67
3        84      83.20  2    16.04 3.288e-04
```

Finally, what is the value obtained by the following command?

```
> mu.hat <- exp(predict(fit2))
> -2 * (sum(dpois(Species, mu.hat, log = TRUE)) - sum(dpois(Species,
+     Species, log = TRUE)))
```

6B Mathematical Biology

A semi-infinite elastic filament lies along the positive x-axis in a viscous fluid. When it is perturbed slightly to the shape $y = h(x,t)$, it evolves according to

$$\zeta\, h_t = -A\, h_{xxxx}\,,$$

where ζ characterises the viscous drag and A the bending stiffness. Motion is forced by boundary conditions

$$h = h_0 \cos(\omega t) \quad \text{and} \quad h_{xx} = 0 \quad \text{at} \quad x = 0, \quad \text{while} \quad h \to 0 \quad \text{as} \quad x \to \infty.$$

Use dimensional analysis to find the characteristic length $\ell(\omega)$ of the disturbance. Show that the steady oscillating solution takes the form

$$h(x,t) = h_0\, \mathrm{Re}\left[e^{\,i\omega t} F(\eta)\right] \quad \text{with} \quad \eta = x/\ell,$$

finding the ordinary differential equation for F.

Find two solutions for F which decay as $x \to \infty$. Without solving explicitly for the amplitudes, show that $h(x,t)$ is the superposition of two travelling waves which decay with increasing x, one with crests moving to the left and one to the right. Which dominates?

7A Dynamical Systems

Let $F : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. State when F is chaotic according to Glendinning's definition.

Prove that if F has a 3-cycle then F^2 has a horseshoe.

[You may assume the Intermediate Value Theorem.]

8C Further Complex Methods

The *Hilbert transform* $\hat{f}$ of a function f is defined by

$$\hat{f}(t) = \frac{1}{\pi}\, \mathcal{P}\!\int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau}\, d\tau,$$

where $\mathcal{P}$ denotes the Cauchy principal value.

Show that the Hilbert transform of $\dfrac{\sin t}{t}$ is $\dfrac{1-\cos t}{t}$.

9B Classical Dynamics

(a) Show that the principal moments of inertia for an infinitesimally thin uniform rectangular sheet of mass M with sides of length a and b (with $b < a$) about its centre of mass are $I_1 = Mb^2/12$, $I_2 = Ma^2/12$ and $I_3 = M(a^2 + b^2)/12$.

(b) Euler's equations governing the angular velocity $(\omega_1, \omega_2, \omega_3)$ of the sheet as viewed in the body frame are

$$I_1 \frac{d\omega_1}{dt} = (I_2 - I_3)\omega_2\omega_3,$$

$$I_2 \frac{d\omega_2}{dt} = (I_3 - I_1)\omega_3\omega_1,$$

and

$$I_3 \frac{d\omega_3}{dt} = (I_1 - I_2)\omega_1\omega_2.$$

A possible solution of these equations is such that the sheet rotates with $\omega_1 = \omega_3 = 0$, and $\omega_2 = \Omega = \text{constant}$.

By linearizing, find the equations governing small motions in the neighbourhood of this solution that have $(\omega_1, \omega_3) \neq 0$. Use these to show that there are solutions corresponding to instability such that ω_1 and ω_3 are both proportional to $\exp(\beta\Omega t)$, with $\beta = \sqrt{(a^2 - b^2)/(a^2 + b^2)}$.

10E Cosmology

The Friedmann and Raychaudhuri equations are respectively

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \qquad \text{and} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right),$$

where ρ is the mass density, P is the pressure, k is the curvature and $\dot{a} \equiv da/dt$ with t the cosmic time. Using conformal time τ (defined by $d\tau = dt/a$) and the equation of state $P = w\rho c^2$, show that these can be rewritten as

$$\frac{kc^2}{\mathcal{H}^2} = \Omega - 1 \qquad \text{and} \qquad 2\frac{d\mathcal{H}}{d\tau} = -(3w+1)\left(\mathcal{H}^2 + kc^2\right),$$

where $\mathcal{H} = a^{-1}da/d\tau$ and the relative density is $\Omega \equiv \rho/\rho_{\text{crit}} = 8\pi G\rho a^2/(3\mathcal{H}^2)$.

Use these relations to derive the following evolution equation for Ω

$$\frac{d\Omega}{d\tau} = (3w+1)\mathcal{H}\Omega(\Omega - 1).$$

For both $w = 0$ and $w = -1$, plot the qualitative evolution of Ω as a function of τ in an expanding universe $\mathcal{H} > 0$ (i.e. include curves initially with $\Omega > 1$ and $\Omega < 1$).

Hence, or otherwise, briefly describe the flatness problem of the standard cosmology and how it can be solved by inflation.

SECTION II

11H Number Theory

Let $N > 1$ be an integer, which is not a square, and let p_k/q_k $(k = 1, 2, \ldots)$ be the convergents to $\sqrt{N}$. Prove that

$$|p_k^2 - q_k^2 N| < 2\sqrt{N} \quad (k = 1, 2, \ldots).$$

Explain briefly how this result can be used to generate a factor base B, and a set of B-numbers which may lead to a factorization of N.

12G Geometry of Group Actions

What does it mean for a subgroup G of the Möbius group to be discrete?

Show that a discrete group necessarily acts properly discontinuously in hyperbolic 3-space.

[You may assume that a discrete subgroup of a matrix group is a closed subset.]

13J Statistical Modelling

Consider the following generalized linear model for responses $y_1, \ldots, y_n$ as a function of explanatory variables $x_1, \ldots, x_n$, where $x_i = (x_{i1}, \ldots, x_{ip})^\top$ for $i = 1, \ldots, n$. The responses are modelled as observed values of independent random variables $Y_1, \ldots, Y_n$, with

$$Y_i \sim \mathrm{ED}(\mu_i, \sigma_i^2), \quad g(\mu_i) = x_i^\top \beta, \quad \sigma_i^2 = \sigma^2 a_i,$$

Here, g is a given link function, β and σ^2 are unknown parameters, and the a_i are treated as known.

[*Hint: recall that we write $Y \sim ED(\mu, \sigma^2)$ to mean that Y has density function of the form*

$$f(y; \mu, \sigma^2) = a(\sigma^2, y) \exp\left\{ \frac{1}{\sigma^2}[\theta(\mu)y - K(\theta(\mu))] \right\}$$

for given functions a and θ.]

[You may use without proof the facts that, for such a random variable Y,

$$E(Y) = K'(\theta(\mu)), \quad \mathrm{var}(Y) = \sigma^2 K''(\theta(\mu)) \equiv \sigma^2 V(\mu).]$$

Show that the score vector and Fisher information matrix have entries:

$$U_j(\beta) = \sum_{i=1}^n \frac{(y_i - \mu_i)x_{ij}}{\sigma_i^2 V(\mu_i) g'(\mu_i)}, \qquad j = 1, \ldots, p,$$

and

$$i_{jk}(\beta) = \sum_{i=1}^n \frac{x_{ij}x_{ik}}{\sigma_i^2 V(\mu_i)(g'(\mu_i))^2}, \qquad j, k = 1, \ldots, p.$$

How do these expressions simplify when the canonical link is used?

Explain briefly how these two expressions can be used to obtain the maximum likelihood estimate $\hat{\beta}$ for β.

14A Dynamical Systems

Explain the difference between a *hyperbolic* and a *nonhyperbolic* fixed point $\mathbf{x}_0$ for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$.

Consider the system in $\mathbb{R}^2$, where μ is a real parameter,

$$\begin{aligned} \dot{x} &= x(\mu - x + y^2), \\ \dot{y} &= y(1 - x - y^2). \end{aligned}$$

Show that the fixed point $(\mu, 0)$ has a bifurcation when $\mu = 1$, while the fixed points $(0, \pm 1)$ have a bifurcation when $\mu = -1$.

[The fixed point at $(0, -1)$ should not be considered further.]

Analyse each of the bifurcations at $(\mu, 0)$ and $(0, 1)$ in turn as follows. Make a change of variable of the form $\mathbf{X} = \mathbf{x} - \mathbf{x}_0(\mu)$, $\nu = \mu - \mu_0$. Identify the (non-extended) stable and centre linear subspaces at the bifurcation in terms of X and Y. By finding the leading-order approximation to the extended centre manifold, construct the evolution equation on the extended centre manifold, and determine the type of bifurcation. Sketch the local bifurcation diagram, showing which fixed points are stable.

[*Hint: the leading-order approximation to the extended centre manifold of the bifurcation at* $(0, 1)$ *is* $Y = aX$ *for some coefficient* a.]

Show that there is another fixed point in $y > 0$ for $\mu < 1$, and that this fixed point connects the two bifurcations.

15B Classical Dynamics

(a) A Hamiltonian system with n degrees of freedom has Hamiltonian $H = H(\mathbf{p}, \mathbf{q})$, where the coordinates $\mathbf{q} = (q_1, q_2, q_3, \ldots, q_n)$ and the momenta $\mathbf{p} = (p_1, p_2, p_3, \ldots, p_n)$ respectively.

Show from Hamilton's equations that when H does not depend on time explicitly, for any function $F = F(\mathbf{p}, \mathbf{q})$,

$$\frac{dF}{dt} = [F, H]\,,$$

where $[F, H]$ denotes the Poisson bracket.

For a system of N interacting vortices

$$H(\mathbf{p}, \mathbf{q}) = -\frac{\kappa}{4} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \ln\left[(p_i - p_j)^2 + (q_i - q_j)^2\right],$$

where κ is a constant. Show that the quantity defined by

$$F = \sum_{j=1}^{N} (q_j^2 + p_j^2)$$

is a constant of the motion.

(b) The action for a Hamiltonian system with one degree of freedom with $H = H(p, q)$ for which the motion is periodic is

$$I = \frac{1}{2\pi} \oint p(H, q) dq.$$

Show without assuming any specific form for H that the period of the motion T is given by

$$\frac{2\pi}{T} = \frac{dH}{dI}.$$

Suppose now that the system has a parameter that is allowed to vary slowly with time. Explain briefly what is meant by the statement that the action is an adiabatic invariant. Suppose that when this parameter is fixed, $H = 0$ when $I = 0$. Deduce that, if T decreases on an orbit with any I when the parameter is slowly varied, then H increases.

16G Logic and Set Theory

Prove Hartog's Lemma that for any set x there is an ordinal α which cannot be mapped injectively into x.

Now assume the Axiom of Choice. Prove Zorn's Lemma and the Well-ordering Principle.

[If you appeal to a fixed point theorem then you should state it clearly.]

17F Graph Theory

For $s \geqslant 2$, let $R(s)$ be the least integer n such that for every 2-colouring of the edges of K_n there is a monochromatic K_s. Prove that $R(s)$ exists.

For any $k \geqslant 1$ and $s_1, \ldots, s_k \geqslant 2$, define the Ramsey number $R_k(s_1, \ldots, s_k)$, and prove that it exists.

Show that, whenever the positive integers are partitioned into finitely many classes, some class contains x, y, z with $x + y = z$.

[*Hint: given a finite colouring of the positive integers, induce a colouring of the pairs of positive integers by giving the pair* ij $(i < j)$ *the colour of* $j - i$.]

18H Galois Theory

Let $L = \mathbf{C}(z)$ be the function field in one variable, $n > 0$ an integer, and $\zeta_n = e^{2\pi i/n}$.

Define $\sigma, \tau : L \to L$ by the formulae

$$(\sigma f)(z) = f(\zeta_n z), \qquad (\tau f)(z) = f(1/z),$$

and let $G = \langle \sigma, \tau \rangle$ be the group generated by σ and τ.

(i) Find $w \in \mathbf{C}(z)$ such that $L^G = \mathbf{C}(w)$.

[You must justify your answer, stating clearly any theorems you use.]

(ii) Suppose n is an odd prime. Determine the subgroups of G and the corresponding intermediate subfields M, with $\mathbf{C}(w) \subseteq M \subseteq L$.

State which intermediate subfields M are Galois extensions of $\mathbf{C}(w)$, and for these extensions determine the Galois group.

19G Representation Theory

(a) Let A be a normal subgroup of a finite group G, and let V be an irreducible representation of G. Show that either V restricted to A is isotypic (a sum of copies of one irreducible representation of A), or else V is induced from an irreducible representation of some proper subgroup of G.

(b) Using (a), show that every (complex) irreducible representation of a p-group is induced from a 1-dimensional representation of some subgroup.

[You may assume that a nonabelian p-group G has an abelian normal subgroup A which is not contained in the centre of G.]

20G Number Fields

(a) Explain what is meant by an integral basis of an algebraic number field. Specify such a basis for the quadratic field $k = \mathbb{Q}(\sqrt{2})$.

(b) Let $K = \mathbb{Q}(\alpha)$ with $\alpha = \sqrt[4]{2}$, a fourth root of 2. Write an element θ of K as

$$\theta = a + b\alpha + c\alpha^2 + d\alpha^3$$

with $a, b, c, d \in \mathbb{Q}$. By computing the relative traces $T_{K/k}(\theta)$ and $T_{K/k}(\alpha\theta)$, show that if θ is an algebraic integer of K, then $2a$, $2b$, $2c$ and $4d$ are rational integers. By further computing the relative norm $N_{K/k}(\theta)$, show that

$$a^2 + 2c^2 - 4bd \quad \text{and} \quad 2ac - b^2 - 2d^2$$

are rational integers. Deduce that $1, \alpha, \alpha^2, \alpha^3$ is an integral basis of K.

21F Algebraic Topology

Let X and Y be topological spaces.

(i) Show that a map $f\colon X \to Y$ is a homotopy equivalence if there exist maps $g, h\colon Y \to X$ such that $fg \simeq 1_Y$ and $hf \simeq 1_X$. More generally, show that a map $f\colon X \to Y$ is a homotopy equivalence if there exist maps $g, h\colon Y \to X$ such that fg and hf are homotopy equivalences.

(ii) Suppose that $\tilde{X}$ and $\tilde{Y}$ are universal covering spaces of the path-connected, locally path-connected spaces X and Y. Using path-lifting properties, show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$.

22F Linear Analysis

Let H be a Hilbert space. Show that if V is a closed subspace of H then any $f \in H$ can be written as $f = v + w$ with $v \in V$ and $w \perp V$.

Suppose $U : H \to H$ is unitary (that is to say $UU^* = U^*U = I$). Let

$$A_n f = \frac{1}{n} \sum_{k=0}^{n-1} U^k f$$

and consider

$$X = \{g - Ug \,:\, g \in H\}.$$

(i) Show that U is an isometry and $\|A_n\| \leqslant 1$.

(ii) Show that X is a subspace of H and $A_n f \to 0$ as $n \to \infty$ whenever $f \in X$.

(iii) Let V be the closure of X. Show that $A_n v \to 0$ as $n \to \infty$ whenever $v \in V$.

(iv) Show that, if $w \perp X$, then $Uw = w$. Deduce that, if $w \perp V$, then $Uw = w$.

(v) If $f \in H$ show that there is a $w \in H$ such that $A_n f \to w$ as $n \to \infty$.

23H Riemann Surfaces

Let Λ be a lattice in $\mathbb{C}$ generated by 1 and τ, where Im $\tau > 0$. The Weierstrass function $\wp$ is the unique meromorphic Λ-periodic function on $\mathbb{C}$, such that the only poles of $\wp$ are at points of Λ and $\wp(z) - 1/z^2 \to 0$ as $z \to 0$.

Show that $\wp$ is an even function. Find all the zeroes of $\wp'$.

Suppose that a is a complex number such that $2a \notin \Lambda$. Show that the function

$$h(z) = (\wp(z-a) - \wp(z+a))(\wp(z) - \wp(a))^2 - \wp'(z)\wp'(a)$$

has no poles in $\mathbb{C} \setminus \Lambda$. By considering the Laurent expansion of h at $z = 0$, or otherwise, deduce that h is constant.

[General properties of meromorphic doubly-periodic functions may be used without proof if accurately stated.]

24H Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface.

(a) In the case where S is compact, define the *Euler characteristic* χ and *genus* g of S.

(b) Define the notion of *geodesic curvature* k_g for regular curves $\gamma : I \to S$. When is $k_g = 0$? State the *Global Gauss–Bonnet Theorem* (including boundary term).

(c) Let $S = \mathbb{S}^2$ (the standard 2-sphere), and suppose $\gamma \subset \mathbb{S}^2$ is a simple closed regular curve such that $\mathbb{S}^2 \setminus \gamma$ is the union of two distinct connected components with equal areas. Can γ have everywhere strictly positive or everywhere strictly negative geodesic curvature?

(d) Prove or disprove the following statement: if S is connected with Gaussian curvature $K = 1$ identically, then S is a subset of a sphere of radius 1.

25J Probability and Measure

(i) A *stepfunction* is any function s on $\mathbb{R}$ which can be written in the form

$$s(x) = \sum_{k=1}^{n} c_k I_{(a_k, b_k]}(x), \quad x \in \mathbb{R},$$

where a_k, b_k, c_k are real numbers, with $a_k < b_k$ for all k. Show that the set of all stepfunctions is dense in $L^1(\mathbb{R}, \mathcal{B}, \mu)$. Here, $\mathcal{B}$ denotes the Borel σ-algebra, and μ denotes Lebesgue measure.

[You may use without proof the fact that, for any Borel set B of finite measure, and any $\varepsilon > 0$, there exists a finite union of intervals A such that $\mu(A \triangle B) < \varepsilon$.]

(ii) Show that the Fourier transform

$$\hat{s}(t) = \int_{\mathbb{R}} s(x) e^{itx} \, dx$$

of a stepfunction has the property that $\hat{s}(t) \to 0$ as $|t| \to \infty$.

(iii) Deduce that the Fourier transform of any integrable function has the same property.

26I Applied Probability

On a hot summer night, opening my window brings some relief. This attracts hordes of mosquitoes who manage to negotiate a dense window net. But, luckily, I have a mosquito trapping device in my room.

Assume the mosquitoes arrive in a Poisson process at rate λ; afterwards they wander around for independent and identically distributed random times with a finite mean $\mathbb{E}S$, where S denotes the random wandering time of a mosquito, and finally are trapped by the device.

(a) Identify a mathematical model, which was introduced in the course, for the number of mosquitoes present in the room at times $t \geqslant 0$.

(b) Calculate the distribution of $Q(t)$ in terms of λ and the tail probabilities $\mathbb{P}(S > x)$ of the wandering time S, where $Q(t)$ is the number of mosquitoes in the room at time $t > 0$ (assuming that at the initial time, $Q(0) = 0$).

(c) Write down the distribution for Q^{E}, the number of mosquitoes in the room in equilibrium, in terms of λ and $\mathbb{E}S$.

(d) Instead of waiting for the number of mosquitoes to reach equilibrium, I close the window at time $t > 0$. For $v \geqslant 0$ let $X(t+v)$ be the number of mosquitoes left at time $t+v$, i.e. v time units after closing the window. Calculate the distribution of $X(t+v)$.

(e) Let $V(t)$ be the time needed to trap all mosquitoes in the room after closing the window at time $t > 0$. By considering the event $\{X(t+v) \geqslant 1\}$, or otherwise, compute $\mathbb{P}[V(t) > v]$.

(f) Now suppose that the time t at which I shut the window is very large, so that I can assume that the number of mosquitoes in the room has the distribution of Q^E. Let V^E be the further time needed to trap all mosquitoes in the room. Show that

$$\mathbb{P}\left[V^E > v\right] = 1 - \exp\left(-\lambda \mathbb{E}\left[(S-v)_+\right]\right),$$

where $x_+ \equiv \max(x, 0)$.

27I Principles of Statistics

Define *sufficient statistic*, and state the factorisation criterion for determining whether a statistic is sufficient. Show that a Bayesian posterior distribution depends on the data only through the value of a sufficient statistic.

Given the value μ of an unknown parameter M, observables $X_1, \ldots, X_n$ are independent and identically distributed with distribution $\mathcal{N}(\mu, 1)$. Show that the statistic $\overline{X} := n^{-1} \sum_{i=1}^{n} X_i$ is sufficient for M.

If the prior distribution is $\mathrm{M} \sim \mathcal{N}(0, \tau^2)$, determine the posterior distribution of M and the predictive distribution of $\overline{X}$.

In fact, there are two hypotheses as to the value of M. Under hypothesis H_0, M takes the known value 0; under H_1, M is unknown, with prior distribution $\mathcal{N}(0, \tau^2)$. Explain why the *Bayes factor* for choosing between H_0 and H_1 depends only on $\overline{X}$, and determine its value for data $X_1 = x_1, \ldots, X_n = x_n$.

The frequentist 5%-level test of H_0 against H_1 rejects H_0 when $|\overline{X}| \geqslant 1.96/\sqrt{n}$. What is the Bayes factor for the critical case $|\overline{x}| = 1.96/\sqrt{n}$? How does this behave as $n \to \infty$? Comment on the similarities or differences in behaviour between the frequentist and Bayesian tests.

28J Stochastic Financial Models

(a) Consider a family $(X_n : n \geqslant 0)$ of independent, identically distributed, positive random variables and fix $z_0 > 0$. Define inductively

$$z_{n+1} = z_n X_n, \quad n \geqslant 0.$$

Compute, for $n \in \{1, \ldots, N\}$, the conditional expectation $\mathbb{E}(z_N | z_n)$.

(b) Fix $R \in [0, 1)$. In the setting of part (a), compute also $\mathbb{E}(U(z_N)|z_n)$, where

$$U(x) = x^{1-R}/(1-R), \quad x \geqslant 0.$$

(c) Let U be as in part (b). An investor with wealth $w_0 > 0$ at time 0 wishes to invest it in such a way as to maximise $\mathbb{E}(U(w_N))$ where w_N is the wealth at the start of day N. Let $\alpha \in [0, 1]$ be fixed. On day n, he chooses the proportion $\theta \in [\alpha, 1]$ of his wealth to invest in a single risky asset, so that his wealth at the start of day $n + 1$ will be

$$w_{n+1} = w_n\{\theta X_n + (1 - \theta)(1 + r)\}.$$

Here, $(X_n : n \geqslant 0)$ is as in part (a) and r is the per-period riskless rate of interest. If $V_n(w) = \sup \mathbb{E}(U(w_N)|w_n = w)$ denotes the value function of this optimization problem, show that $V_n(w_n) = a_n U(w_n)$ and give a formula for a_n. Verify that, in the case $\alpha = 1$, your answer is in agreement with part (b).

29I Optimization and Control

State Pontryagin's maximum principle for the controllable dynamical system with state-space $\mathbb{R}^+$, given by

$$\dot{x}_t = b(t, x_t, u_t), \quad t \geqslant 0,$$

where the running costs are given by $c(t, x_t, u_t)$, up to an unconstrained terminal time τ when the state first reaches 0, and there is a terminal cost $C(\tau)$.

A company pays a variable price $p(t)$ per unit time for electrical power, agreed in advance, which depends on the time of day. The company takes on a job at time $t = 0$, which requires a total amount E of electrical energy, but can be processed at a variable level of power consumption $u(t) \in [0, 1]$. If the job is completed by time τ, then the company will receive a reward $R(\tau)$. Thus, it is desired to minimize

$$\int_0^\tau u(t)p(t)dt - R(\tau),$$

subject to

$$\int_0^\tau u(t)dt = E, \quad u(t) \in [0, 1],$$

with $\tau > 0$ unconstrained. Take as state variable the energy x_t still needed at time t to complete the job. Use Pontryagin's maximum principle to show that the optimal control is to process the job on full power or not at all, according as the price $p(t)$ lies below or above a certain threshold value p^*.

Show further that, if τ^* is the completion time for the optimal control, then

$$p^* + \dot{R}(\tau^*) \ = \ p(\tau^*)\,.$$

Consider a case in which p is periodic, with period one day, where day 1 corresponds to the time interval $[0, 2]$, and $p(t) = (t - 1)^2$ during day 1. Suppose also that $R(\tau) = 1/(1+\tau)$ and $E = 1/2$. Determine the total energy cost and the reward associated with the threshold $p^* = 1/4$.

Hence, show that any threshold low enough to carry processing over into day 2 is suboptimal.

Show carefully that the optimal price threshold is given by $p^* = 1/4$.

30C Partial Differential Equations

(i) Define the Fourier transform $\hat{f} = \mathcal{F}(f)$ of a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, and also of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) From your definition, compute the Fourier transform of the distribution $W_t \in \mathcal{S}'(\mathbb{R}^3)$ given by

$$W_t(\psi) \;=< W_t, \psi >= \;\frac{1}{4\pi t}\int_{\|y\|=t} \psi(y)\, d\Sigma(y)$$

for every Schwartz function $\psi \in \mathcal{S}(\mathbb{R}^3)$. Here $d\Sigma(y) = t^2 d\Omega(y)$ is the integration element on the sphere of radius t.

Hence deduce the formula of Kirchoff for the solution of the initial value problem for the wave equation in three space dimensions,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u \;=\; 0,$$

with initial data $u(0,x) = 0$ and $\frac{\partial u}{\partial t}(0,x) = g(x)$, $x \in \mathbb{R}^3$, where $g \in \mathcal{S}(\mathbb{R}^3)$. Explain briefly why the formula is also valid for arbitrary smooth $g \in C^\infty(\mathbb{R}^3)$.

(iii) Show that any C^2 solution of the initial value problem in (ii) is given by the formula derived in (ii) (uniqueness).

(iv) Show that any two C^2 solutions of the initial value problem for

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u \;=\; 0\,,$$

with the same initial data as in (ii), also agree for any $t > 0$.

31A Asymptotic Methods

The Bessel equation of order n is

$$z^2 y'' + zy' + \left(z^2 - n^2\right) y \;=\; 0\,. \tag{1}$$

Here, n is taken to be an integer, with $n \geqslant 0$. The transformation $w(z) \;=\; z^{\frac{1}{2}}\, y(z)$ converts (1) to the form

$$w'' + q(z)\, w \;=\; 0\,, \tag{2}$$

where

$$q(z) \;=\; 1 - \frac{\left(n^2 - \frac{1}{4}\right)}{z^2}\,.$$

Find two linearly independent solutions of the form

$$w \;=\; e^{\,sz} \sum_{k=0}^{\infty} c_k z^{\,\rho - k}\,, \tag{3}$$

where c_k are constants, with $c_0 \neq 0$, and s and ρ are to be determined. Find recurrence relationships for the c_k.

Find the first two terms of two linearly independent Liouville–Green solutions of (2) for $w(z)$ valid in a neighbourhood of $z = \infty$. Relate these solutions to those of the form (3).

32D Principles of Quantum Mechanics

Define the interaction picture for a quantum mechanical system with Schrödinger picture Hamiltonian $H_0 + V(t)$ and explain why either picture gives the same physical predictions. Derive an equation of motion for interaction picture states and use this to show that the probability of a transition from a state $|n\rangle$ at time zero to a state $|m\rangle$ at time t is

$$P(t) \;=\; \frac{1}{\hbar^2}\left| \int_0^t e^{i(E_m - E_n)t'/\hbar} \langle m|V(t')|n\rangle \, dt' \right|^2$$

correct to second order in V, where the initial and final states are orthogonal eigenstates of H_0 with eigenvalues E_n and E_m.

Consider a perturbed harmonic oscillator:

$$H_0 \;=\; \hbar\omega(a^\dagger a + \tfrac{1}{2})\;, \qquad V(t) \;=\; \hbar\lambda(\, a e^{i\nu t} + a^\dagger e^{-i\nu t}\,)$$

with a and $a^\dagger$ annihilation and creation operators (all usual properties may be assumed). Working to order λ^2, find the probability for a transition from an initial state with $E_n = \hbar\omega(n + \frac{1}{2})$ to a final state with $E_m = \hbar\omega(m + \frac{1}{2})$ after time t.

Suppose t becomes large and perturbation theory still applies. Explain why the rate $P(t)/t$ for each allowed transition is sharply peaked, as a function of ν, around $\nu = \omega$.

33E Applications of Quantum Mechanics

Explain why the allowed energies of electrons in a three-dimensional crystal lie in energy bands. What quantum numbers can be used to classify the electron energy eigenstates?

Describe the effect on the energy level structure of adding a small density of impurity atoms randomly to the crystal.

34D Statistical Physics

Show that the Fermi momentum p_F of a gas of N non-interacting electrons in volume V is

$$p_F = \left(3\pi^2\hbar^3 \frac{N}{V}\right)^{1/3} .$$

Consider the electrons to be effectively massless, so that an electron of momentum p has (relativistic) energy cp. Show that the mean energy per electron at zero temperature is $3cp_F/4$.

When a constant external magnetic field of strength B is applied to the electron gas, each electron gets an energy contribution $\pm\mu B$ depending on whether its spin is parallel or antiparallel to the field. Here μ is the magnitude of the magnetic moment of an electron. Calculate the total magnetic moment of the electron gas at zero temperature, assuming μB is much less than cp_F.

35D Electrodynamics

The Maxwell field tensor is given by

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

A general 4-velocity is written as $U^a = \gamma(1, \mathbf{v})$, where $\gamma = (1 - |\mathbf{v}|^2)^{-1/2}$, and $c = 1$. A general 4-current density is written as $J^a = (\rho, \mathbf{j})$, where ρ is the charge density and $\mathbf{j}$ is the 3-current density. Show that

$$F^{ab}U_b = \gamma(\mathbf{E} \cdot \mathbf{v},\ \mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

In the rest frame of a conducting medium, Ohm's law states that $\mathbf{j} = \sigma \mathbf{E}$ where σ is the conductivity. Show that the relativistic generalization to a frame in which the medium moves with uniform velocity $\mathbf{v}$ is

$$J^a - (J^b U_b)U^a = \sigma F^{ab} U_b.$$

Show that this implies

$$\mathbf{j} = \rho \mathbf{v} + \sigma \gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B} - (\mathbf{v} \cdot \mathbf{E})\mathbf{v}).$$

Simplify this formula, given that the charge density vanishes in the rest frame of the medium.

36E General Relativity

A solution of the Einstein equations is given by the metric

$$\mathrm{d}s^2 = -\Big(1 - \frac{2M}{r}\Big)\,\mathrm{d}t^2 + \frac{1}{\Big(1 - \frac{2M}{r}\Big)}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,.$$

For an incoming light ray, with constant θ, ϕ, show that

$$t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|,$$

for some fixed v and find a similar solution for an outgoing light ray. For the outgoing case, assuming $r > 2M$, show that in the far past $\frac{r}{2M} - 1 \propto \exp(\frac{t}{2M})$ and in the far future $r \sim t$.

Obtain the transformed metric after the change of variables $(t, r, \theta, \phi) \rightarrow (v, r, \theta, \phi)$. With coordinates $\hat{t} = v - r, r$ sketch, for fixed θ, ϕ, the trajectories followed by light rays. What is the significance of the line $r = 2M$?

Show that, whatever path an observer with initial $r = r_0 < 2M$ takes, he must reach $r = 0$ in a finite proper time.

37A Fluid Dynamics II

Viscous incompressible fluid of uniform density is extruded axisymmetrically from a thin circular slit of small radius centred at the origin and lying in the plane $z = 0$ in cylindrical polar coordinates r, θ, z. There is no external radial pressure gradient. It is assumed that the fluid forms a thin boundary layer, close to and symmetric about the plane $z = 0$. The layer has thickness $\delta(r) \ll r$. The r-component of the steady Navier–Stokes equations may be approximated by

$$u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} = \nu \frac{\partial^2 u_r}{\partial z^2}.$$

(i) Prove that the quantity (proportional to the flux of radial momentum)

$$\mathcal{F} = \int_{-\infty}^{\infty} u_r^2 \, r \, dz$$

is independent of r.

(ii) Show, by balancing terms in the momentum equation and assuming constancy of $\mathcal{F}$, that there is a similarity solution of the form

$$u_r = -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad \Psi = -A\delta(r) f(\eta), \quad \eta = \frac{z}{\delta(r)}, \quad \delta(r) = Cr,$$

where A, C are constants. Show that for suitable choices of A and C the equation for f takes the form

$$-f'^2 - ff'' = f''';$$

$$f = f'' = 0 \text{ at } \eta = 0; \qquad f' \to 0 \text{ as } \eta \to \infty;$$

$$\int_{-\infty}^{\infty} f_\eta^2 \, d\eta = 1.$$

(iii) Give an inequality connecting $\mathcal{F}$ and ν that ensures that the boundary layer approximation ($\delta \ll r$) is valid. Solve the equation to give a complete solution to the problem for u_r when this inequality holds.

[*Hint:* $\int_{-\infty}^{\infty} \operatorname{sech}^4 x \, dx = 4/3$.]

38B Waves

A layer of rock of shear modulus $\bar{\mu}$ and shear wave speed $\bar{c}_s$ occupies the region $0 \leqslant y \leqslant h$ with a free surface at $y = h$. A second rock having shear modulus μ and shear wave speed $c_s > \bar{c}_s$ occupies $y \leqslant 0$. Show that elastic SH waves of wavenumber k and phase speed c can propagate in the layer with zero disturbance at $y = -\infty$ if $\bar{c}_s < c < c_s$ and c satisfies the dispersion relation

$$\tan\left[kh\sqrt{c^2/\bar{c}_s^2 - 1} \right] = \frac{\mu}{\bar{\mu}} \frac{\sqrt{1 - c^2/c_s^2}}{\sqrt{c^2/\bar{c}_s^2 - 1}} .$$

Show graphically, or otherwise, that this equation has at least one real solution for any value of kh, and determine the smallest value of kh for which the equation has at least two real solutions.

39C Numerical Analysis

Let $A \in \mathbb{R}^{n\times n}$ be a real matrix with n linearly independent eigenvectors. When calculating eigenvalues of A, the sequence $\mathbf{x}^{(k)}$, $k = 0, 1, 2, \ldots$, is generated by the power method $\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}/\|A\mathbf{x}^{(k)}\|$, where $\mathbf{x}^{(0)}$ is a real nonzero vector.

(a) Describe the asymptotic properties of the sequence $\mathbf{x}^{(k)}$, both in the case where the eigenvalues λ_i of A satisfy $|\lambda_i| < |\lambda_n|$, $i = 1, \ldots, n-1$, and in the case where $|\lambda_i| < |\lambda_{n-1}| = |\lambda_n|$, $i = 1, \ldots, n-2$. In the latter case explain how the (possibly complex-valued) eigenvalues λ_{n-1}, λ_n and their corresponding eigenvectors can be determined.

(b) Let $n = 3$, and suppose that, for a large k, we obtain the vectors

$$\mathbf{y}_k = \mathbf{x}_k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_{k+1} = A\mathbf{x}_k = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{y}_{k+2} = A^2\mathbf{x}_k = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

Find two eigenvalues of A and their corresponding eigenvectors.

END OF PAPER