MATHEMATICAL TRIPOS Part II

Thursday 5 June 2008 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Prove that, for all $x \geqslant 2$, we have

$$\sum_{p \leqslant x} \frac{1}{p} > \log\log x - \frac{1}{2}.$$

[You may assume that, for $0 < u < 1$,

$$-\log(1-u) - u < \frac{u^2}{2(1-u)}.]$$

2F Topics in Analysis

(a) State the Baire category theorem in its closed sets version.

(b) Let $f_n : \mathbf{R} \to \mathbf{R}$ be a continuous function for each $n = 1, 2, 3, \ldots$ and suppose that there is a function $f : \mathbf{R} \to \mathbf{R}$ such that $f_n(x) \to f(x)$ for each $x \in \mathbf{R}$. Prove that for each $\epsilon > 0$, there exists an integer N_0 and a non-empty open interval $I \subset \mathbf{R}$ such that $|f_n(x) - f(x)| \leqslant \epsilon$ for all $n \geqslant N_0$ and $x \in I$.

[*Hint: consider, for* $N = 1, 2, 3, \ldots$, *the sets*

$$Q_N = \{x \in \mathbf{R} : |f_n(x) - f_m(x)| \leqslant \epsilon : \forall n, m \geqslant N\}.]$$

3G Geometry of Group Actions

Let $\dim_H$ denote the Hausdorff dimension of a set in $\mathbb{R}^n$. Prove that if $\dim_H(F) < 1$ then F is totally disconnected.

[You may assume that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map then

$$\dim_H(f(F)) \leqslant \dim_H(F).]$$

4G Coding and Cryptography

Define the Hamming code $h : \mathbb{F}_2^4 \to \mathbb{F}_2^7$ and prove that the minimum distance between two distinct code words is 3. Explain how the Hamming code allows one error to be corrected.

A new code $c : \mathbb{F}_2^4 \to \mathbb{F}_2^8$ is obtained by using the Hamming code for the first 7 bits and taking the last bit as a check digit on the previous 7. Find the minimum distance between two distinct code words for this code. How many errors can this code detect? How many errors can it correct?

5J Statistical Modelling

Consider the linear model $Y = X\beta + \varepsilon$. Here, Y is an n-dimensional vector of observations, X is a known $n \times p$ matrix, β is an unknown p-dimensional parameter, and $\varepsilon \sim N_n(0, \sigma^2 I)$, with σ^2 unknown. Assume that X has full rank and that $p \ll n$. Suppose that we are interested in checking the assumption $\varepsilon \sim N_n(0, \sigma^2 I)$. Let $\hat{Y} = X\hat{\beta}$, where $\hat{\beta}$ is the maximum likelihood estimate of β. Write in terms of X an expression for the projection matrix $P = (p_{ij} : 1 \leqslant i, j \leqslant n)$ which appears in the maximum likelihood equation $\hat{Y} = X\hat{\beta} = PY$.

Find the distribution of $\hat{\varepsilon} = Y - \hat{Y}$, and show that, in general, the components of $\hat{\varepsilon}$ are not independent.

A standard procedure used to check our assumption on ε is to check whether the studentized fitted residuals

$$\hat{\eta}_i = \frac{\hat{\varepsilon}_i}{\tilde{\sigma}\sqrt{1 - p_{ii}}}, \quad i = 1, \ldots, n,$$

look like a random sample from an $N(0,1)$ distribution. Here,

$$\tilde{\sigma}^2 = \frac{1}{n-p}||Y - X\hat{\beta}||^2.$$

Say, briefly, how you might do this in `R`.

This procedure appears to ignore the dependence between the components of $\hat{\varepsilon}$ noted above. What feature of the given set-up makes this reasonable?

6B Mathematical Biology

An allosteric enzyme E reacts with a substrate S to produce a product P according to the mechanism

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_1 \overset{k_2}{\rightharpoonup} E + P$$

$$S + C_1 \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} C_2 \overset{k_4}{\rightharpoonup} C_1 + P,$$

where C_1 and C_2 are enzyme-substrate complexes. With lowercase letters denoting concentrations, write down a system of differential equations based on the Law of Mass Action which model this reaction mechanism.

The initial conditions are $s = s_0$, $e = e_0$, $c_1 = c_2 = p = 0$. Using $u = s/s_0$, $v_i = c_i/e_0$, $\tau = k_1 e_0 t$ and $\epsilon = e_0/s_0$, show that the nondimensional reaction mechanism reduces to

$$\frac{du}{d\tau} = f(u, v_1, v_2) \quad \text{and} \quad \epsilon \frac{dv_i}{d\tau} = g_i(u, v_1, v_2) \quad \text{for} \quad i = 1, 2,$$

finding expressions for f, g_1 and g_2.

7A Dynamical Systems

State the normal-form equations for (i) a saddle-node bifurcation, (ii) a transcritical bifurcation and (iii) a pitchfork bifurcation, for a one-dimensional map $x_{n+1} = F(x_n; \mu)$.

Consider a period-doubling bifurcation of the form

$$x_{n+1} = -x_n + \alpha + \beta x_n + \gamma x_n^2 + \delta x_n^3 + O(x_n^4),$$

where $x_n = O(\mu^{1/2})$, $\alpha, \beta = O(\mu)$, and $\gamma, \delta = O(1)$ as $\mu \to 0$. Show that

$$X_{n+2} = X_n + \hat{\mu} X_n - A X_n^3 + O(X_n^4),$$

where $X_n = x_n - \frac{1}{2}\alpha$, and the parameters $\hat{\mu}$ and A are to be identified in terms of α, β, γ and δ. Deduce the condition for the bifurcation to be supercritical.

8C Further Complex Methods

What is the effect of the Möbius transformation $z \to \dfrac{z}{z-1}$ on the points $z=0$, $z=\infty$ and $z=1$?

By considering

$$(z-1)^{-a} P \left\{ \begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z(z-1)^{-1} \\ 1-c & c-b & b-a & \end{matrix} \right\},$$

or otherwise, show that $(z-1)^{-a}F(a, c-b; c; z(z-1)^{-1})$ is a branch of the P-function

$$P \left\{ \begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix} \right\}.$$

Give a linearly independent branch.

9E Classical Dynamics

Writing $\mathbf{x} = (p_1, p_2, p_3, \ldots, p_n, q_1, q_2, q_3, \ldots, q_n)$, Hamilton's equations may be written in the form

$$\dot{\mathbf{x}} = \mathbf{J}\frac{\partial H}{\partial \mathbf{x}},$$

where the $2n \times 2n$ matrix

$$\mathbf{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

and I and 0 denote the $n \times n$ unit and zero matrices respectively.

Explain what is meant by the statement that the transformation $\mathbf{x} \to \mathbf{y}$,

$$(p_1, p_2, p_3, \ldots, p_n, q_1, q_2, q_3, \ldots, q_n) \to (P_1, P_2, P_3, \ldots, P_n, Q_1, Q_2, Q_3, \ldots, Q_n),$$

is canonical, and show that the condition for this is that

$$\mathbf{J} = \mathcal{J}\mathbf{J}\mathcal{J}^T,$$

where $\mathcal{J}$ is the Jacobian matrix with elements

$$\mathcal{J}_{ij} = \frac{\partial y_i}{\partial x_j}.$$

Use this condition to show that for a system with $n = 1$ the transformation given by

$$P = p + 2q, \quad Q = \frac{1}{2}q - \frac{1}{4}p$$

is canonical.

10E Cosmology

The energy density ϵ and pressure P of photons in the early universe is given by

$$\epsilon = \frac{4\sigma}{c}T^4, \qquad P = \frac{1}{3}\epsilon,$$

where σ is the Stefan–Boltzmann constant. By using the first law of thermodynamics $dE = TdS - PdV + \mu dN$, deduce that the entropy differential dS can be expressed in the form

$$dS = \frac{16\sigma}{3c}d(T^3V).$$

With the third law, show that the entropy density is given by $s = (16\sigma/3c)T^3$.

While particle interaction rates Γ remain much greater than the Hubble parameter H, justify why entropy will be conserved during the expansion of the universe. Hence, in the early universe (radiation domination) show that the temperature $T \propto a^{-1}$ where $a(t)$ is the scale factor of the universe, and show that the Hubble parameter $H \propto T^2$.

SECTION II

11H Number Theory

State the reciprocity law for the Jacobi symbol.

Let a be an odd integer > 1, which is not a square. Prove that there exists a positive integer n such that $n \equiv 1 \bmod 4$ and

$$\left(\frac{n}{a}\right) = -1.$$

Prove further that there exist infinitely many prime numbers p such that

$$\left(\frac{a}{p}\right) = -1.$$

12F Topics in Analysis

(a) State Liouville's theorem on approximation of algebraic numbers by rationals.

(b) Consider the continued fraction expression

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

in which the coefficients a_n are all positive integers forming an unbounded set. Let $\frac{p_n}{q_n}$ be the nth convergent. Prove that

$$\left| x - \frac{p_n}{q_n} \right| \leqslant \frac{1}{q_n q_{n+1}}$$

and use this inequality together with Liouville's theorem to deduce that x^2 is irrational.

[You may assume without proof that, for $n = 1, 2, 3, \ldots$,

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}.]$$

13B Mathematical Biology

Consider the activator-inhibitor system in the fast-inhibitor limit

$$u_t \;=\; D\,u_{xx} - u\,(u-r)(u-1) - \rho\,(v-u),$$

$$0 \;=\; v_{xx} - (v-u),$$

where D is small, $0 < r < 1$ and $0 < \rho < 1$.

Examine the linear stability of the state $u = v = 0$ using perturbations of the form $\exp(ikx + \sigma t)$. Sketch the growth-rate σ as a function of the wavenumber k. Find the growth-rate of the most unstable wave, and so determine the boundary in the r-ρ parameter plane which separates stable and unstable modes.

Show that the system is unchanged under the transformation $u \to 1-u$, $v \to 1-v$ and $r \to 1-r$. Hence write down the equation for the boundary between stable and unstable modes of the state $u = v = 1$.

14A Dynamical Systems

Define the Poincaré index of a simple closed curve, not necessarily a trajectory, and the Poincaré index of an isolated fixed point $\mathbf{x_0}$ for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. State the Poincaré index of a periodic orbit.

Consider the system

$$\dot{x} = y + ax - bx^3,$$
$$\dot{y} = x^3 - x,$$

where a and b are constants and $a \neq 0$.

(a) Find and classify the fixed points, and state their Poincaré indices.

(b) By considering a suitable function $H(x, y)$, show that any periodic orbit Γ satisfies

$$\oint_\Gamma (x - x^3)(ax - bx^3)dt = 0,$$

where $x(t)$ is evaluated along the orbit.

(c) Deduce that if $b/a < 1$ then the second-order differential equation

$$\ddot{x} - (a - 3bx^2)\dot{x} + x - x^3 = 0$$

has no periodic solutions.

15E Cosmology

Small density perturbations $\delta_{\mathbf{k}}(t)$ in pressureless matter inside the cosmological horizon obey the following Fourier evolution equation

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} - 4\pi G\bar{\rho}_{\mathrm{c}}\delta_{\mathbf{k}} = 0,$$

where $\bar{\rho}_{\mathrm{c}}$ is the average background density of the pressureless gravitating matter and $\mathbf{k}$ is the comoving wavevector.

(i) Seek power law solutions $\delta_{\mathbf{k}} \propto t^{\beta}$ (β constant) during the matter-dominated epoch ($t_{\mathrm{eq}} < t < t_0$) to find the approximate solution

$$\delta_{\mathbf{k}}(t) = A(\mathbf{k})\left(\frac{t}{t_{\mathrm{eq}}}\right)^{2/3} + B(\mathbf{k})\left(\frac{t}{t_{\mathrm{eq}}}\right)^{-1}, \qquad t \gg t_{\mathrm{eq}}$$

where A, B are functions of $\mathbf{k}$ only and t_{eq} is the time of equal matter-radiation.

By considering the behaviour of the scalefactor a and the relative density $\bar{\rho}_{\mathrm{c}}/\bar{\rho}_{\mathrm{total}}$, show that early in the radiation era ($t \ll t_{\mathrm{eq}}$) there is effectively no significant perturbation growth in $\delta_{\mathbf{k}}$ on sub-horizon scales.

(ii) For a given wavenumber $k = |\mathbf{k}|$, show that the time t_{H} at which this mode crosses inside the horizon, i.e., $ct_{\mathrm{H}} \approx 2\pi a(t_{\mathrm{H}})/k$, is given by

$$\frac{t_{\mathrm{H}}}{t_0} \approx \begin{cases} \left(\frac{k_0}{k}\right)^3, & t_{\mathrm{H}} \gg t_{\mathrm{eq}}, \\ (1+z_{\mathrm{eq}})^{-1/2}\left(\frac{k_0}{k}\right)^2, & t_{\mathrm{H}} \ll t_{\mathrm{eq}}, \end{cases}$$

where $k_0 \equiv 2\pi/(ct_0)$, and the equal matter-radiation redshift is given by $1 + z_{\mathrm{eq}} = (t_0/t_{\mathrm{eq}})^{2/3}$.

Assume that primordial perturbations from inflation are scale-invariant with a constant amplitude as they cross the Hubble radius given by $\langle|\delta_{\mathbf{k}}(t_{\mathrm{H}})|^2\rangle \approx V^{-1}A/k^3$, where A is a constant and V is a large volume. Use the results of (i) to project these perturbations forward to t_0, and show that the power spectrum for perturbations today will be given approximately by

$$P(k) \;\equiv\; V\langle|\delta_{\mathbf{k}}(t_0)|^2\rangle \;\approx\; \frac{A}{k_0^4} \times \begin{cases} k, & k < k_{\mathrm{eq}}, \\ k_{\mathrm{eq}}\left(\frac{k_{\mathrm{eq}}}{k}\right)^3, & k > k_{\mathrm{eq}}. \end{cases}$$

16G Logic and Set Theory

What is a transitive set? Show that if x is transitive then so are the union $\bigcup x$ and the power set Px of x. If $\bigcup x$ is transitive, is x transitive? If Px is transitive, is x transitive? Justify your answers.

What is the transitive closure of a set? Show that any set x has a transitive closure $TC(x)$.

Suppose that x has rank α. What is the rank of Px? What is the rank of $TC(x)$?

[You may use standard properties of rank.]

17F Graph Theory

Define the chromatic polynomial $p_G(t)$ of a graph G. Show that if G has n vertices and m edges then

$$p_G(t) = a_n t^n - a_{n-1} t^{n-1} + a_{n-2} t^{n-2} - \ldots + (-1)^n a_0,$$

where $a_n = 1$ and $a_{n-1} = m$ and $a_i \geqslant 0$ for all $0 \leqslant i \leqslant n$. [You may assume the deletion–contraction relation, provided it is clearly stated.]

Show that if G is a tree on n vertices then $p_G(t) = t(t-1)^{n-1}$. Does the converse hold?

[*Hint: if G is disconnected, how is the chromatic polynomial of G related to the chromatic polynomials of its components?*]

Show that if G is a graph on n vertices with the same chromatic polynomial as $T_r(n)$ (the Turán graph on n vertices with r vertex classes) then G must be isomorphic to $T_r(n)$.

18H Galois Theory

Let L/K be a field extension.

(a) State what it means for $\alpha \in L$ to be algebraic over K, and define its degree $\deg_K(\alpha)$. Show that if $\deg_K(\alpha)$ is odd, then $K(\alpha) = K(\alpha^2)$.

[You may assume any standard results.]

Show directly from the definitions that if α, $\beta \in L$ are algebraic over K, then so too is $\alpha + \beta$.

(b) State what it means for $\alpha \in L$ to be separable over K, and for the extension L/K to be separable.

Give an example of an inseparable extension L/K.

Show that an extension L/K is separable if L is a finite field.

19G Representation Theory

Let V_2 denote the irreducible representation $\mathrm{Sym}^2(\mathbb{C}^2)$ of $SU(2)$; thus V_2 has dimension 3. Compute the character of the representation $\mathrm{Sym}^n(V_2)$ of $SU(2)$ for any $n \geqslant 0$. Compute the dimension of the invariants $\mathrm{Sym}^n(V_2)^{SU(2)}$, meaning the subspace of $\mathrm{Sym}^n(V_2)$ where $SU(2)$ acts trivially.

Hence, or otherwise, show that the ring of complex polynomials in three variables x, y, z which are invariant under the action of $SO(3)$ is a polynomial ring. Find a generator for this polynomial ring.

20F Algebraic Topology

Let X be the quotient space obtained by identifying one pair of antipodal points on S^2. Using the Mayer–Vietoris exact sequence, calculate the homology groups and the Betti numbers of X.

21F Linear Analysis

State and prove the Stone–Weierstrass theorem for real-valued functions. You may assume that the function $x \mapsto |x|$ can be uniformly approximated by polynomials on any interval $[-k, k]$.

Suppose that $0 < a < b < 1$. Let $\mathcal{F}$ be the set of functions which can be uniformly approximated on $[a, b]$ by polynomials with *integer* coefficients. By making appropriate use of the identity

$$\frac{1}{2} = \frac{x}{1-(1-2x)} = \sum_{n=0}^{\infty} x(1-2x)^n,$$

or otherwise, show that $\mathcal{F} = \mathcal{C}([a, b])$.

Is it true that every continuous function on $[0, b]$ can be uniformly approximated by polynomials with *integer* coefficients?

22H Riemann Surfaces

Define the degree of a non-constant holomorphic map between compact connected Riemann surfaces and state the Riemann–Hurwitz formula.

Show that there exists a compact connected Riemann surface of any genus $g \geqslant 0$.

[You may use without proof any foundational results about holomorphic maps and complex algebraic curves from the course, provided that these are accurately stated. You may also assume that if $h(s)$ is a non-constant complex polynomial without repeated roots then the algebraic curve $C = \{(s,t) \in \mathbb{C}^2 : t^2 - h(s) = 0\}$ is path connected.]

23H Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface.

(a) Define the *Gauss Map*, *principal curvatures* k_i, *Gaussian curvature* K and *mean curvature* H. State the *Theorema Egregium*.

(b) Define what is meant for S to be *minimal*. Prove that if S is minimal, then $K \leqslant 0$. Give an example of a minimal surface whose Gaussian curvature is not identically 0, justifying your answer.

(c) Does there exist a *compact* minimal surface $S \subset \mathbb{R}^3$? Justify your answer.

24J Probability and Measure

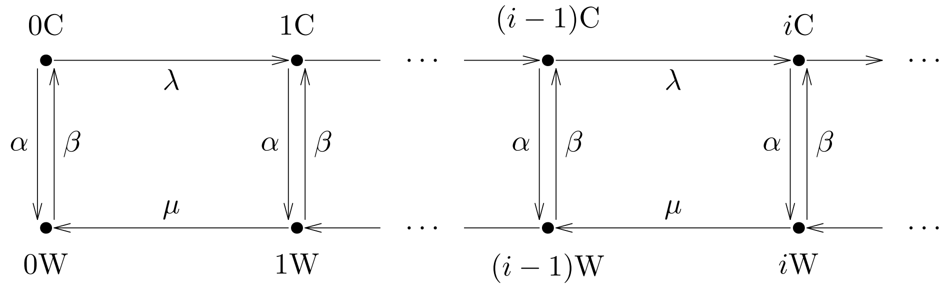
(i) What does it mean to say that a sequence of random variables (X_n) converges *in probability* to X? What does it mean to say that the sequence (X_n) converges *in distribution* to X? Prove that if $X_n \to X$ in probability, then $X_n \to X$ in distribution.

(ii) What does it mean to say that a sequence of random variables (X_n) is *uniformly integrable*? Show that, if (X_n) is uniformly integrable and $X_n \to X$ in distribution, then $\mathbb{E}(X_n) \to \mathbb{E}(X)$.

[Standard results from the course may be used without proof if clearly stated.]

25I Applied Probability

Let (X_t) be an irreducible continuous-time Markov chain with countably many states. What does it mean to say the chain is (i) positive recurrent, (ii) null recurrent? Consider the chain (X_t) with the arrow diagram below.



In this question we analyse the existence of equilibrium probabilities $\pi_{i\mathrm{C}}$ and $\pi_{i\mathrm{W}}$ of the chain (X_t) being in state iC or iW, $i = 0, 1, \ldots$, and the impact of this fact on positive and null recurrence of the chain.

(a) Write down the invariance equations $\pi Q = 0$ and check that they have the form

$$\pi_{0C} = \frac{\beta}{\lambda + \alpha}\, \pi_{0\mathrm{W}},$$

$$\big(\pi_{1\mathrm{C}}, \pi_{1\mathrm{W}}\big) = \frac{\beta \pi_{0\mathrm{W}}}{\lambda + \alpha} \left(\frac{\lambda(\mu + \beta)}{\mu(\lambda + \alpha)}, \frac{\lambda}{\mu} \right),$$

$$\big(\pi_{(i+1)\mathrm{C}}, \pi_{(i+1)\mathrm{W}}\big) = \big(\pi_{i\mathrm{C}}, \pi_{i\mathrm{W}}\big) B, \qquad i = 1, 2, \ldots,$$

where B is a 2×2 recursion matrix:

$$B = \begin{pmatrix} \dfrac{\lambda\mu - \beta\alpha}{\mu(\lambda + \alpha)} & -\dfrac{\alpha}{\mu} \\[2ex] \dfrac{\beta(\beta + \mu)}{\mu(\lambda + \alpha)} & \dfrac{\beta + \mu}{\mu} \end{pmatrix}.$$

(b) Verify that the row vector $\big(\pi_{1\mathrm{C}}, \pi_{1\mathrm{W}}\big)$ is an eigenvector of B with the eigenvalue θ where

$$\theta = \frac{\lambda\,(\mu + \beta)}{\mu\,(\lambda + \alpha)}.$$

Hence, specify the form of equilibrium probabilities $\pi_{i\mathrm{C}}$ and $\pi_{i\mathrm{W}}$ and conclude that the chain (X_t) is positive recurrent if and only if $\mu\alpha > \lambda\beta$.

26I Principles of Statistics

Define the notion of *exponential family (EF)*, and show that, for data arising as a random sample of size n from an exponential family, there exists a sufficient statistic whose dimension stays bounded as $n \to \infty$.

The log-density of a normal distribution $\mathcal{N}(\mu, v)$ can be expressed in the form

$$\log p(x \mid \boldsymbol{\phi}) \;=\; \phi_1\, x + \phi_2\, x^2 - k(\boldsymbol{\phi})$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2)$ is the value of an unknown parameter $\Phi = (\Phi_1, \Phi_2)$. Determine the function k, and the natural parameter-space $\mathbb{F}$. What is the *mean-value parameter* $\mathrm{H} = (\mathrm{H}_1, \mathrm{H}_2)$ in terms of Φ?

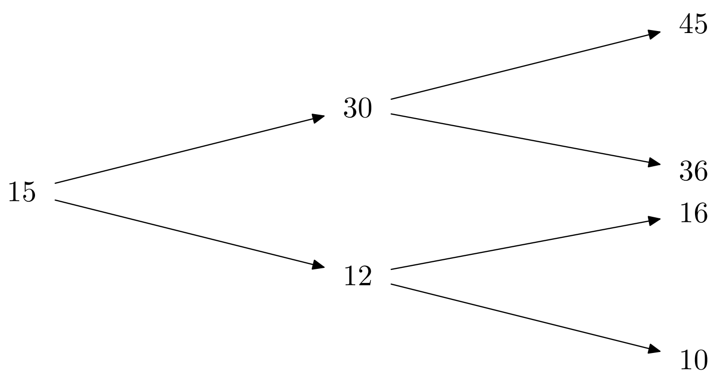
Determine the maximum likelihood estimator $\widehat{\Phi}_1$ of Φ_1 based on a random sample $(X_1, \ldots, X_n)$, and give its asymptotic distribution for $n \to \infty$.

How would these answers be affected if the variance of X were known to have value v_0?

27J Stochastic Financial Models

Consider a vector of asset prices evolving over time $\bar{S} = (S_t^0, S_t^1, \ldots, S_t^d)_{t \in \{0,1,\ldots,T\}}$. The asset price S^0 is assumed constant over time. In this context, explain what is an arbitrage and prove that the existence of an equivalent martingale measure implies no-arbitrage.

Suppose that over two periods a stock price moves on a binomial tree



Assume riskless rate $r = 1/4$. Determine the equivalent martingale measure. [No proof is required.]

Sell an American put with strike 15 and expiry 2 at its no-arbitrage price, which you should determine.

Verify that the buyer of the option should use his early exercise right if the first period is bad.

Assume that the first period is bad, and that the buyer forgets to exercise. How much risk-free profit can you lock in?

28I Optimization and Control

Let Q be a positive-definite symmetric $m \times m$ matrix. Show that a non-negative quadratic form on $\mathbb{R}^d \times \mathbb{R}^m$ of the form

$$c(x,a) = x^T Rx + x^T S^T a + a^T Sx + a^T Qa, \quad x \in \mathbb{R}^d, \quad a \in \mathbb{R}^m,$$

is minimized over a, for each x, with value $x^T(R - S^T Q^{-1} S)x$, by taking $a = Kx$, where $K = -Q^{-1}S$.

Consider for $k \leqslant n$ the controllable stochastic linear system in $\mathbb{R}^d$

$$X_{j+1} = AX_j + BU_j + \varepsilon_{j+1}, \quad j = k, k+1, \ldots, n-1,$$

starting from $X_k = x$ at time k, where the control variables U_j take values in $\mathbb{R}^m$, and where $\varepsilon_{k+1}, \ldots, \varepsilon_n$ are independent, zero-mean random variables, with $\mathrm{var}(\varepsilon_j) = N_j$. Here, A, B and N_j are, respectively, $d \times d$, $d \times m$ and $d \times d$ matrices. Assume that a cost $c(X_j, U_j)$ is incurred at each time $j = k, \ldots, n-1$ and that a final cost $C(X_n) = X_n^T \Pi_0 X_n$ is incurred at time n. Here, Π_0 is a given non-negative-definite symmetric matrix. It is desired to minimize, over the set of all controls u, the total expected cost $V^u(k,x)$. Write down the optimality equation for the infimal cost function $V(k,x)$.

Hence, show that $V(k,x)$ has the form

$$V(k,x) \;=\; x^T \Pi_{n-k} x + \gamma_k$$

for some non-negative-definite symmetric matrix Π_{n-k} and some real constant γ_k. Show how to compute the matrix Π_{n-k} and constant γ_k and how to determine an optimal control.

29C Partial Differential Equations

Let $C^\infty_{per} = \{u \in C^\infty(\mathbb{R}) : u(x+2\pi) = u(x)\}$ be the space of smooth 2π-periodic functions of one variable.

(i) For $f \in C^\infty_{per}$ show that there exists a unique $u_f \in C^\infty_{per}$ such that

$$-\frac{d^2 u_f}{dx^2} + u_f = f.$$

(ii) Show that $I_f[u_f + \phi] > I_f[u_f]$ for every $\phi \in C^\infty_{per}$ which is not identically zero, where $I_f : C^\infty_{per} \to \mathbb{R}$ is defined by

$$I_f[u] = \frac{1}{2}\int_{-\pi}^{+\pi} \left[\left(\frac{\partial u}{\partial x}\right)^2 + u^2 - 2f(x)\,u \right] dx.$$

(iii) Show that the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = f(x),$$

with initial data $u(0,x) = u_0(x) \in C^\infty_{per}$ has, for $t > 0$, a smooth solution $u(t,x)$ such that $u(t,\cdot) \in C^\infty_{per}$ for each fixed $t > 0$. Give a representation of this solution as a Fourier series in x. Calculate $\lim_{t\to+\infty} u(t,x)$ and comment on your answer in relation to (i).

(iv) Show that $I_f[u(t,\cdot)] \leqslant I_f[u(s,\cdot)]$ for $t > s > 0$, and that $I_f[u(t,\cdot)] \to I_f[u_f]$ as $t \to +\infty$.

30A Asymptotic Methods

Describe how the leading-order approximation may be found by the method of stationary phase of

$$I(\lambda) = \int_a^b f(t) \exp\big(i\lambda\, g(t)\big)\, dt,$$

for $\lambda \gg 1$, where λ, f and g are real. You should consider the cases for which:

(a) $g'(t)$ has one simple zero at $t = t_0$, where $a < t_0 < b$;

(b) $g'(t)$ has more than one simple zero in the region $a < t < b$; and

(c) $g'(t)$ has only a simple zero at $t = b$.

What is the order of magnitude of $I(\lambda)$ if $g'(t)$ is non zero for $a \leqslant t \leqslant b$?

Use the method of stationary phase to find the leading-order approximation for $\lambda \gg 1$ to

$$J(\lambda) = \int_0^1 \sin\big(\lambda\left(t^3 - t\right)\big)\, dt.$$

[*Hint:*

$$\int_{-\infty}^{\infty} \exp\left(iu^2\right) du = \sqrt{\pi} e^{i\pi/4}\,.]$$

31C Integrable Systems

Let $U(\lambda)$ and $V(\lambda)$ be matrix-valued functions of (x, y) depending on the auxiliary parameter λ. Consider a system of linear PDEs

$$\frac{\partial}{\partial x}\Phi = U(\lambda)\Phi, \quad \frac{\partial}{\partial y}\Phi = V(\lambda)\Phi \tag{1}$$

where Φ is a column vector whose components depend on (x, y, λ). Derive the zero curvature representation as the compatibility conditions for this system.

Assume that

$$U(\lambda) = -\begin{pmatrix} u_x & 0 & \lambda \\ 1 & -u_x & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V(\lambda) = -\begin{pmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^{u} \\ \lambda^{-1}e^{u} & 0 & 0 \end{pmatrix}$$

and show that (1) is compatible if the function $u = u(x, y)$ satisfies the PDE

$$\frac{\partial^2 u}{\partial x \partial y} = F(u) \tag{2}$$

for some $F(u)$ which should be determined.

Show that the transformation

$$(x, y) \longrightarrow (cx, c^{-1}y), \qquad c \in \mathbb{R} \setminus \{0\}$$

forms a symmetry group of the PDE (2) and find the vector field generating this group.

Find the ODE characterising the group-invariant solutions of (2).

32D Principles of Quantum Mechanics

Explain, in a few lines, how the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are used to represent angular momentum operators with respect to basis states $|\uparrow\rangle$ and $|\downarrow\rangle$ corresponding to spin up and spin down along the 3-axis. You should state clearly which properties of the matrices correspond to general features of angular momentum and which are specific to spin half.

Consider two spin-half particles labelled A and B, each with its spin operators and spin eigenstates. Find the matrix representation of

$$\boldsymbol{\sigma}^{(\mathrm{A})} \cdot \boldsymbol{\sigma}^{(\mathrm{B})} \;=\; \sigma_1^{(\mathrm{A})}\sigma_1^{(\mathrm{B})} + \sigma_2^{(\mathrm{A})}\sigma_2^{(\mathrm{B})} + \sigma_3^{(\mathrm{A})}\sigma_3^{(\mathrm{B})}$$

with respect to a basis of two-particle states $|\uparrow\rangle_{\mathrm{A}}|\uparrow\rangle_{\mathrm{B}}\,,\ |\downarrow\rangle_{\mathrm{A}}|\uparrow\rangle_{\mathrm{B}}\,,\ |\uparrow\rangle_{\mathrm{A}}|\downarrow\rangle_{\mathrm{B}}\,,\ |\downarrow\rangle_{\mathrm{A}}|\downarrow\rangle_{\mathrm{B}}$. Show that the eigenvalues of the matrix are $1, 1, 1, -3$ and find the eigenvectors.

What is the behaviour of each eigenvector under interchange of A and B? If the particles are identical, and there are no other relevant degrees of freedom, which of the two-particle states are allowed?

By relating $(\boldsymbol{\sigma}^{(\mathrm{A})} + \boldsymbol{\sigma}^{(\mathrm{B})})^2$ to the operator discussed above, show that your findings are consistent with standard results for addition of angular momentum.

33E Applications of Quantum Mechanics

Consider the body-centred cuboidal lattice L with lattice points $(n_1 a, n_2 a, n_3 b)$ and $((n_1 + \frac{1}{2})a, (n_2 + \frac{1}{2})a, (n_3 + \frac{1}{2})b)$, where a and b are positive and n_1, n_2 and n_3 take all possible integer values. Find the reciprocal lattice $\widetilde{L}$ and describe its geometrical form. Calculate the volumes of the unit cells of the lattices L and $\widetilde{L}$.

Find the reciprocal lattice vector associated with the lattice planes parallel to the plane containing the points $(0, 0, b)$, $(0, a, b)$, $(\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}b)$, $(a, 0, 0)$ and $(a, a, 0)$. Deduce the allowed Bragg scattering angles of X-rays off these planes, assuming that $b = \frac{4}{3}a$ and that the X-rays have wavelength $\lambda = \frac{1}{2}a$.

34E Statistical Physics

Derive the following two relations:

$$T\,dS \;=\; C_p\,dT \;-\; T\left.\frac{\partial V}{\partial T}\right|_p dp$$

and

$$T\,dS \;=\; C_V\,dT \;+\; T\left.\frac{\partial p}{\partial T}\right|_V dV.$$

[You may use any standard Maxwell relation without proving it.]

Experimentalists very seldom measure C_V directly; they measure C_p and use thermodynamics to extract C_V. Use your results from the first part of this question to find a formula for $C_p - C_V$ in terms of the easily measured quantities

$$\alpha \;=\; \frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_p$$

(the volume coefficient of expansion) and

$$\kappa \;=\; -\frac{1}{V}\left.\frac{\partial V}{\partial p}\right|_T$$

(the isothermal compressibility).

35D Electrodynamics

The retarded scalar potential $\varphi(t, \mathbf{x})$ produced by a charge distribution $\rho(t, \mathbf{x})$ is given by

$$\varphi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_\Omega d^3x' \, \frac{\rho(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

where Ω denotes all 3-space. Describe briefly and qualitatively the physics underlying this formula.

Write the integrand in the formula above as a 1-dimensional integral over a new time coordinate τ. Next consider a special source, a point charge q moving along a trajectory $\mathbf{x} = \mathbf{x}_0(t)$ so that

$$\rho(t, \mathbf{x}) = q\delta^{(3)}(\mathbf{x} - \mathbf{x}_0(t)),$$

where $\delta^{(3)}(\mathbf{x})$ denotes the 3-dimensional delta function. By reversing the order of integration, or otherwise, obtain the Liénard–Wiechert potential

$$\varphi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{q}{R - \mathbf{v} \cdot \mathbf{R}},$$

where $\mathbf{v}$ and $\mathbf{R}$ are to be determined.

Write down the corresponding formula for the vector potential $\mathbf{A}(t, \mathbf{x})$.

36A Fluid Dynamics II

Show that, in cylindrical polar co-ordinates, the streamfunction $\psi(r,\phi)$ for the velocity $\mathbf{u} = (u_r(r,\phi), u_\phi(r,\phi), 0)$ and vorticity $(0, 0, \omega(r,\phi))$ of two-dimensional Stokes flow of incompressible fluid satisfies the equations

$$\mathbf{u} = \left(\frac{1}{r}\frac{\partial\psi}{\partial\phi}, -\frac{\partial\psi}{\partial r}, 0\right), \qquad \nabla^2\omega = -\nabla^4\psi = 0.$$

Show also that the pressure $p(r,\phi)$ satisfies $\nabla^2 p = 0$.

A stationary rigid circular cylinder of radius a occupies the region $r \leqslant a$. The flow around the cylinder tends at large distances to a simple shear flow, with velocity given in cartesian coordinates (x, y, z) by $\mathbf{u} = (\Gamma y, 0, 0)$. Inertial forces may be neglected.

By solving the equation for ψ in cylindrical polars, determine the flow field everywhere. Determine the torque on the cylinder per unit length in z.

[*Hint: in cylindrical polars*

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2}.$$

The off-diagonal component of the rate-of-strain tensor is given by

$$e_{r\phi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right).]$$

37B Waves

The real function $\phi(x,t)$ satisfies the Klein–Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} - \phi\,, \quad -\infty < x < \infty,\ t \geqslant 0\,.$$

Find the dispersion relation for disturbances of wavenumber k and deduce their phase and group velocities.

Suppose that at $t = 0$

$$\phi(x,0) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial t}(x,0) = e^{-|x|}\,.$$

Use Fourier transforms to find an integral expression for $\phi(x,t)$ when $t > 0$.

Use the method of stationary phase to find $\phi(Vt,t)$ for $t \to \infty$ for fixed $0 < V < 1$. What can be said if $V > 1$?

[*Hint: you may assume that*

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}\,, \quad \mathrm{Re}(a) > 0\,.]$$

38C Numerical Analysis

(a) A numerical method for solving the ordinary differential equation

$$y'(t) = f(t,y), \qquad t \in [0,T], \qquad y(0) = y_0,$$

generates for every $h > 0$ a sequence $\{y_n\}$, where y_n is an approximation to $y(t_n)$ and $t_n = nh$. Explain what is meant by the convergence of the method.

(b) Prove from first principles that if the function f is sufficiently smooth and satisfies the Lipschitz condition

$$|f(t,x) - f(t,y)| \leqslant \lambda |x-y|, \qquad x,y \in \mathbb{R}, \qquad t \in [0,T]\,,$$

for some $\lambda > 0$, then the trapezoidal rule

$$y_{n+1} = y_n + \frac{1}{2} h \left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right]$$

converges.

END OF PAPER