UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Wednesday 4 June 2008 9.00 to 12.00

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

What does it mean for a positive definite quadratic form with integer coefficients to be reduced?

Show that there are precisely three reduced forms of this type with discriminant equal to -23.

Which odd primes are properly represented by some positive definite binary quadratic form (with integer coefficients) of discriminant -23?

2F Topics in Analysis

(a) State Brouwer's fixed point theorem in the plane and prove that the statement is equivalent to non-existence of a continuous retraction of the closed disk D to its boundary ∂D.

(b) Use Brouwer's fixed point theorem to prove that there is a complex number z in the closed unit disc such that $z^6 - z^5 + 2z^2 + 6z + 1 = 0$.

3G Geometry of Group Actions

State a theorem classifying lattices in $\mathbb{R}^2$. Define a frieze group.

Show there is a frieze group which is isomorphic to $\mathbb{Z}$ but is not generated by a translation, and draw a picture whose symmetries are this group.

4G Coding and Cryptography

Describe briefly the Shannon–Fano and Huffman binary codes for a finite alphabet. Find examples of such codes for the alphabet $\mathcal{A} = \{a, b, c, d\}$ when the four letters are taken with probabilities $0.4, 0.3, 0.2$ and 0.1 respectively.

5J Statistical Modelling

Suppose that we want to estimate the angles α, β and γ (in radians, say) of the triangle ABC, based on a single independent measurement of the angle at each corner. Suppose that the error in measuring each angle is normally distributed with mean zero and variance σ^2. Thus, we model our measurements y_A, y_B, y_C as the observed values of random variables

$$Y_A = \alpha + \varepsilon_A, \quad Y_B = \beta + \varepsilon_B, \quad Y_C = \gamma + \varepsilon_C,$$

where $\varepsilon_A, \varepsilon_B, \varepsilon_C$ are independent, each with distribution $N(0, \sigma^2)$. Find the maximum likelihood estimate of α based on these measurements.

Can the assumption that $\varepsilon_A, \varepsilon_B, \varepsilon_C \sim N(0, \sigma^2)$ be criticized? Why or why not?

6B Mathematical Biology

The population dynamics of a species is governed by the discrete model

$$N_{t+1} = f(N_t) = N_t \exp\left[r\left(1 - \frac{N_t}{K}\right)\right],$$

where r and K are positive constants.

Determine the steady states and their eigenvalues. Show that a period-doubling bifurcation occurs at $r = 2$.

Show graphically that the maximum possible population after $t = 0$ is

$$N_{max} = f(K/r).$$

7A Dynamical Systems

Explain the difference between a *stationary bifurcation* and an *oscillatory bifurcation* for a fixed point $\mathbf{x}_0$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x};\mu)$ in $\mathbb{R}^n$ with a real parameter μ.

The normal form of a Hopf bifurcation in polar coordinates is

$$\begin{aligned}\dot{r} &= \mu r - ar^3 + O(r^5),\\ \dot{\theta} &= \omega + c\mu - br^2 + O(r^4),\end{aligned}$$

where a, b, c and ω are constants, $a \neq 0$, and $\omega > 0$. Sketch the phase plane near the bifurcation for each of the cases (i) $\mu < 0 < a$, (ii) $0 < \mu, a$, (iii) $\mu, a < 0$ and (iv) $a < 0 < \mu$.

Let R be the radius and T the period of the limit cycle when one exists. Sketch how R varies with μ for the case when the limit cycle is subcritical. Find the leading-order approximation to $dT/d\mu$ for $|\mu| \ll 1$.

8C Further Complex Methods

The Beta function is defined for $\mathrm{Re}\, z > 0$ by

$$\mathrm{B}(z,q) = \int_0^1 t^{q-1}(1-t)^{z-1}dt \qquad (\mathrm{Re}\, q > 0)$$

and by analytic continuation elsewhere in the complex z-plane.

Show that

$$\left(\frac{z+q}{z}\right)\mathrm{B}(z+1,q) = \mathrm{B}(z,q)$$

and explain how this result can be used to obtain the analytic continuation of $\mathrm{B}(z,q)$. Hence show that $\mathrm{B}(z,q)$ is analytic except for simple poles and find the residues at the poles.

9A Classical Dynamics

A system of N particles $i = 1, 2, 3, \ldots, N$, with mass m_i, moves around a circle of radius a. The angle between the radius to particle i and a fixed reference radius is θ_i. The interaction potential for the system is

$$V = \frac{1}{2}k\sum_{j=1}^{N}(\theta_{j+1} - \theta_j)^2,$$

where k is a constant and $\theta_{N+1} = \theta_1 + 2\pi$.

The Lagrangian for the system is

$$L = \frac{1}{2}a^2\sum_{j=1}^{N} m_j\dot{\theta}_j^2 - V.$$

Write down the equation of motion for particle i and show that the system is in equilibrium when the particles are equally spaced around the circle.

Show further that the system always has a normal mode of oscillation with zero frequency. What is the form of the motion associated with this?

Find all the frequencies and modes of oscillation when $N = 2$, $m_1 = km/a^2$ and $m_2 = 2km/a^2$, where m is a constant.

10E Cosmology

A spherically-symmetric star obeys the pressure-support equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2},$$

where $P(r)$ is the pressure at a distance r from the centre, $\rho(r)$ is the density, and $m(r)$ is the mass within a sphere of radius r. Show that this implies

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2\rho.$$

Propose and justify appropriate boundary conditions for the pressure $P(r)$ at the centre of the star ($r = 0$) and at its outer edge $r = R$.

Show that the function

$$F(r) = P(r) + \frac{Gm^2}{8\pi r^4}$$

is a decreasing function of r. Deduce that the central pressure $P_c \equiv P(0)$ satisfies

$$P_c > \frac{GM^2}{8\pi R^4},$$

where $M \equiv m(R)$ is the mass of the star.

SECTION II

11F Topics in Analysis

Let $L : C([0,1]) \to C([0,1])$ be an operator satisfying the conditions

(i) $Lf \geqslant 0$ for any $f \in C([0,1])$ with $f \geqslant 0$,

(ii) $L(af + bg) = aLf + bLg$ for any $f, g \in C([0,1])$ and $a, b \in \mathbf{R}$ and

(iii) $Z_f \subseteq Z_{Lf}$ for any $f \in C([0,1])$, where Z_f denotes the set of zeros of f.

Prove that there exists a function $h \in C([0,1])$ with $h \geqslant 0$ such that $Lf = hf$ for every $f \in C([0,1])$.

12G Coding and Cryptography

Describe the Rabin cipher with modulus N, explaining how it can be deciphered by the intended recipient and why it is difficult for an interceptor to decipher it.

The Bursars' Committee decides to communicate using Rabin ciphers to maintain confidentiality. The secretary of the committee encrypts a message, thought of as a positive integer m, using the Rabin cipher with modulus N (with $0 < m < N$) and publishes both the encrypted message and the modulus. A foolish bursar deciphers this message to read it but then encrypts it again using a Rabin cipher with a different modulus N' (with $m < N'$) and publishes the newly encrypted message and N'. The president of CUSU, who happens to be a talented mathematician, knows that this has happened. Explain how the president can work out what the original message was using the two different encrypted versions.

Can the president of CUSU also decipher other messages sent out by the Bursars' Committee?

13B Mathematical Biology

Consider the nonlinear equation describing the invasion of a population $u(x,t)$

$$u_t \;=\; m\,u_{xx} + f(u), \tag{1}$$

with $m > 0$, $f(u) = -u\,(u-r)(u-1)$ and $0 < r < 1$ a constant.

(a) Considering time-dependent spatially homogeneous solutions, show that there are two stable and one unstable uniform steady states.

(b) In the case $r = \frac{1}{2}$, find the stationary 'front' which has

$$u \to 1 \text{ as } x \to -\infty \quad \text{and} \quad u \to 0 \text{ as } x \to \infty. \tag{2}$$

[*Hint:* $f(u) = F'(u)$ *where* $F(u) = -\frac{1}{4}u^2(1-u)^2 + \frac{1}{6}(r-\frac{1}{2})u^2(2u-3)$.]

(c) Now consider travelling-wave solutions to (1) of the form $u(x,t) = U(z)$ where $z = x - vt$. Show that U satisfies an equation of the form

$$m\,\ddot{U} + v\,\dot{U} \;=\; -V'(U),$$

where $(\dot{\ }) \equiv \dfrac{d}{dz}(\ \ )$ and $(\ \ )' \equiv \dfrac{d}{dU}(\ \ )$.

Sketch the form of $V(U)$ for $r = \frac{1}{2}$, $r > \frac{1}{2}$ and $r < \frac{1}{2}$. Using conditions (2), show that

$$v\int_{-\infty}^{\infty} \dot{U}^2\,dz \;=\; F(1) - F(0).$$

Deduce how the sign of the travelling-wave velocity v depends on r.

14C Further Complex Methods

(i) The function f is defined by

$$f(z) = \int_C t^{z-1} dt\,,$$

where C is the circle $|t| = r$, described anti-clockwise starting on the positive real axis and where the value of t^{z-1} at each point on C is determined by analytic continuation along C with $\arg t = 0$ at the starting point. Verify by direct integration that f is an entire function, the values of which depend on r.

(ii) The function J is defined by

$$J(z) = \int_\gamma e^t (t^2 - 1)^z dt,$$

where γ is a figure of eight, starting at $t = 0$, looping anti-clockwise round $t = 1$ and returning to $t = 0$, then looping clockwise round $t = -1$ and returning again to $t = 0$. The value of $(t^2 - 1)^z$ is determined by analytic continuation along γ with $\arg(t^2 - 1) = -\pi$ at the start. Show that, for $\operatorname{Re} z > -1$,

$$J(z) = -2i \sin \pi z \, I(z),$$

where

$$I(z) = \int_{-1}^{1} e^t (t^2 - 1)^z dt.$$

Explain how this provides the analytic continuation of $I(z)$. Classify the singular points of the analytically continued function, commenting on the points $z = 0, 1, \ldots$.

Explain briefly why the analytic continuation could not be obtained by this method if γ were replaced by the circle $|t| = 2$.

15B Classical Dynamics

A particle of mass m, charge e and position vector $\mathbf{r} = (x_1, x_2, x_3) \equiv \mathbf{q}$ moves in a magnetic field whose vector potential is $\mathbf{A}$. Its Hamiltonian is given by

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2m}\left(\mathbf{p} - e\frac{\mathbf{A}}{c}\right)^2 .$$

Write down Hamilton's equations and use them to derive the equations of motion for the charged particle.

Define the Poisson bracket $[F, G]$ for general $F(\mathbf{p}, \mathbf{q})$ and $G(\mathbf{p}, \mathbf{q})$. Show that for motion governed by the above Hamiltonian

$$[m\dot{x}_i, x_j] = -\delta_{ij}, \qquad \text{and} \qquad [m\dot{x}_i, m\dot{x}_j] = \frac{e}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right).$$

Consider the vector potential to be given by $\mathbf{A} = (0, 0, F(r))$, where $r = \sqrt{x_1^2 + x_2^2}$. Use Hamilton's equations to show that p_3 is constant and that circular motion at radius r with angular frequency Ω is possible provided that

$$\Omega^2 = -\left(p_3 - \frac{eF}{c}\right)\frac{e}{m^2 c r}\frac{dF}{dr}.$$

16G Logic and Set Theory

(i) State the Completeness Theorem and the Compactness Theorem for the predicate calculus.

(ii) Show that if a theory has arbitrarily large finite models then it has an infinite model. Deduce that there is no first order theory whose models are just the finite fields of characteristic 2. Show that the theory of infinite fields of characteristic 2 does not have a finite axiomatisation.

(iii) Let $\mathcal{T}$ be the collection of closed terms in some first order language $\mathcal{L}$. Suppose that $\exists x.\phi(x)$ is a provable sentence of $\mathcal{L}$ with ϕ quantifier-free. Show that the set of sentences $\{\neg\phi(t) : t \in \mathcal{T}\}$ is inconsistent.

[*Hint: consider the minimal substructure of a model.*]

Deduce that there are $t_1, t_2, \ldots, t_n$ in $\mathcal{T}$ such that $\phi(t_1) \vee \phi(t_2) \vee \cdots \vee \phi(t_n)$ is provable.

17F Graph Theory

Prove that every graph G on $n \geqslant 3$ vertices with minimum degree $\delta(G) \geqslant \frac{n}{2}$ is Hamiltonian. For each $n \geqslant 3$, give an example to show that this result does not remain true if we weaken the condition to $\delta(G) \geqslant \frac{n}{2} - 1$ (for n even) or $\delta(G) \geqslant \frac{n-1}{2}$ (for n odd).

For any graph G, let G_k denote the graph formed by adding k new vertices to G, all joined to each other and to all vertices of G. By considering G_1, show that if G is a graph on $n \geqslant 3$ vertices with $\delta(G) \geqslant \frac{n-1}{2}$ then G has a Hamilton path (a path passing through all the vertices of G).

For each positive integer k, exhibit a connected graph G such that G_k is not Hamiltonian. Is this still possible if we replace 'connected' with '2-connected'?

18H Galois Theory

(i) Let K be a field, $\theta \in K$, and $n > 0$ not divisible by the characteristic. Suppose that K contains a primitive nth root of unity. Show that the splitting field of $x^n - \theta$ has cyclic Galois group.

(ii) Let L/K be a Galois extension of fields and ζ_n denote a primitive nth root of unity in some extension of L, where n is not divisible by the characteristic. Show that $\mathrm{Aut}(L(\zeta_n)/K(\zeta_n))$ is a subgroup of $\mathrm{Aut}(L/K)$.

(iii) Determine the minimal polynomial of a primitive 6th root of unity ζ_6 over $\mathbf{Q}$.

Compute the Galois group of $x^6 + 3 \in \mathbf{Q}[x]$.

19G Representation Theory

A finite group G of order 360 has conjugacy classes $C_1 = \{1\}$, $C_2, \ldots, C_7$ of sizes 1, 45, 40, 40, 90, 72, 72. The values of four of its irreducible characters are given in the following table.

C_1	C_2	C_3	C_4	C_5	C_6	C_7
5	1	2	-1	-1	0	0
8	0	-1	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$
8	0	-1	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
10	-2	1	1	0	0	0

Complete the character table.

[*Hint: it will not suffice just to use orthogonality of characters.*]

Deduce that the group G is simple.

20G Number Fields

(a) Factorise the ideals [2], [3] and [5] in the ring of integers $\mathcal{O}_K$ of the field $K = \mathbb{Q}(\sqrt{30})$. Using Minkowski's bound

$$\frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|d_K|},$$

determine the ideal class group of K.

[*Hint: it might be helpful to notice that* $\frac{3}{2} = N_{K/\mathbb{Q}}(\alpha)$ *for some* $\alpha \in K$.]

(b) Find the fundamental unit of K and determine all solutions of the equations $x^2 - 30y^2 = \pm 5$ in integers $x, y \in \mathbb{Z}$. Prove that there are in fact no solutions of $x^2 - 30y^2 = 5$ in integers $x, y \in \mathbb{Z}$.

21F Algebraic Topology

Prove the Borsuk–Ulam theorem in dimension 2: there is no map $f\colon S^2 \to S^1$ such that $f(-x) = -f(x)$ for every $x \in S^2$. Deduce that S^2 is not homeomorphic to any subset of $\mathbb{R}^2$.

22F Linear Analysis

State and prove the principle of uniform boundedness.

[You may assume the Baire category theorem.]

Suppose that X, Y and Z are Banach spaces. Suppose that

$$F : X \times Y \to Z$$

is linear and continuous in each variable separately, that is to say that, if y is fixed,

$$F(\cdot, y) : X \to Z$$

is a continuous linear map and, if x is fixed,

$$F(x, \cdot) : Y \to Z$$

is a continuous linear map. Show that there exists an M such that

$$\|F(x,y)\|_Z \leqslant M\|x\|_X\|y\|_Y$$

for all $x \in X$, $y \in Y$. Deduce that F is continuous.

Suppose X, Y, Z and W are Banach spaces. Suppose that

$$G : X \times Y \times W \to Z$$

is linear and continuous in each variable separately. Does it follow that G is continuous? Give reasons.

Suppose that X, Y and Z are Banach spaces. Suppose that

$$H : X \times Y \to Z$$

is continuous in each variable separately. Does it follow that H is continuous? Give reasons.

23H Riemann Surfaces

Explain what is meant by a divisor D on a compact connected Riemann surface S. Explain briefly what is meant by a canonical divisor. Define the degree of D and the notion of linear equivalence between divisors. If two divisors on S have the same degree must they be linearly equivalent? Give a proof or a counterexample as appropriate, stating accurately any auxiliary results that you require.

Define $\ell(D)$ for a divisor D, and state the Riemann–Roch theorem. Deduce that the dimension of the space of holomorphic differentials is determined by the genus g of S and that the same is true for the degree of a canonical divisor. Show further that if $g = 2$ then S admits a non-constant meromorphic function with at most two poles (counting with multiplicities).

[General properties of meromorphic functions and meromorphic differentials on S may be used without proof if clearly stated.]

24H Differential Geometry

(a) For a regular curve in $\mathbb{R}^3$, define *curvature* and *torsion* and state the *Frenet formulas.*

(b) State and prove the isoperimetric inequality for domains $\Omega \subset \mathbb{R}^2$ with compact closure and C^1 boundary $\partial\Omega$.

[You may assume Wirtinger's inequality.]

(c) Let $\gamma : I \to \mathbb{R}^2$ be a *closed* plane regular curve such that γ is contained in a disc of radius r. Show that there exists $s \in I$ such that $|k(s)| \geqslant r^{-1}$, where $k(s)$ denotes the signed curvature. Show by explicit example that the assumption of closedness is necessary.

25J Probability and Measure

Explain what is meant by a *simple function* on a measurable space $(S, \mathcal{S})$.

Let $(S, \mathcal{S}, \mu)$ be a finite measure space and let $f : S \to \mathbb{R}$ be a non-negative Borel measurable function. State the definition of the integral of f with respect to μ.

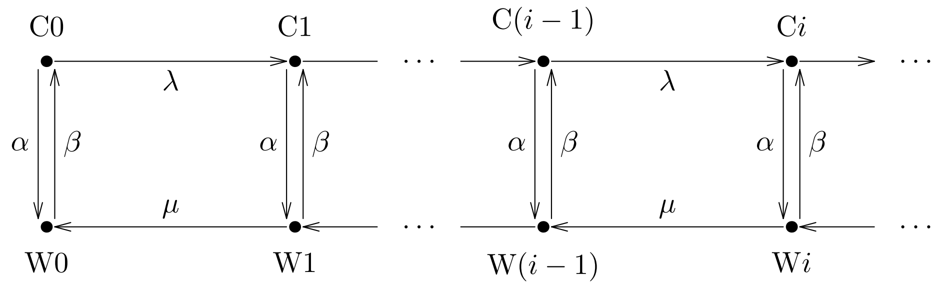
Prove that, for any sequence of simple functions (g_n) such that $0 \leqslant g_n(x) \uparrow f(x)$ for all $x \in S$, we have

$$\int g_n d\mu \uparrow \int f d\mu.$$

State and prove the Monotone Convergence Theorem for finite measure spaces.

26I Applied Probability

Consider a continuous-time Markov chain (X_t) given by the diagram below.



We will assume that the rates α, β, λ and μ are all positive.

(a) Is the chain (X_t) irreducible?

(b) Write down the standard equations for the hitting probabilities

$$h_{\mathrm{C}i} = \mathbb{P}_{\mathrm{C}i}\big(\text{hit W0}\big), \quad i \geqslant 0,$$

and

$$h_{\mathrm{W}i} = \mathbb{P}_{\mathrm{W}i}\big(\text{hit W0}\big), \quad i \geqslant 1.$$

Explain how to identify the probabilities $h_{\mathrm{C}i}$ and $h_{\mathrm{W}i}$ among the solutions to these equations.

[You should state the theorem you use but its proof is not required.]

(c) Set $h^{(i)} = \begin{pmatrix} h_{\mathrm{C}i} \\ h_{\mathrm{W}i} \end{pmatrix}$ and find a matrix A such that

$$h^{(i)} = Ah^{(i-1)}, \quad i = 1, 2, \ldots.$$

The recursion matrix A has a 'standard' eigenvalue and a 'standard' eigenvector that do not depend on the transition rates: what are they and why are they always present?

(d) Calculate the second eigenvalue ϑ of the matrix A, and the corresponding eigenvector, in the form $\begin{pmatrix} b \\ 1 \end{pmatrix}$, where $b > 0$.

(e) Suppose the second eigenvalue ϑ is $\geqslant 1$. What can you say about $h_{\mathrm{C}i}$ and $h_{\mathrm{W}i}$? Is the chain (X_t) transient or recurrent? Justify your answer.

(f) Now assume the opposite: the second eigenvalue ϑ is < 1. Check that in this case $b < 1$. Is the chain transient or recurrent under this condition?

(g) Finally, specify, by means of inequalities between the parameters α, β, λ and μ, when the chain (X_t) is recurrent and when it is transient.

27I Principles of Statistics

Under hypothesis H_i $(i = 0, 1)$, a real-valued observable X, taking values in $\mathcal{X}$, has density function $p_i(\cdot)$. Define the *Type I error* α and the *Type II error* β of a test $\phi : \mathcal{X} \to [0, 1]$ of the null hypothesis H_0 against the alternative hypothesis H_1. What are the *size* and *power* of the test in terms of α and β?

Show that, for $0 < c < \infty$, ϕ minimises $c\alpha + \beta$ among all possible tests if and only if it satisfies

$$p_1(x) > c\,p_0(x) \Rightarrow \phi(x) = 1,$$
$$p_1(x) < c\,p_0(x) \Rightarrow \phi(x) = 0.$$

What does this imply about the admissibility of such a test?

Given the value θ of a parameter variable $\Theta \in [0, 1)$, the observable X has density function

$$p(x \mid \theta) = \frac{2(x - \theta)}{(1 - \theta)^2} \qquad (\theta \leqslant x \leqslant 1).$$

For fixed $\theta \in (0, 1)$, describe all the likelihood ratio tests of $H_0 : \Theta = 0$ against $H_\theta : \Theta = \theta$.

For fixed $k \in (0, 1)$, let ϕ_k be the test that rejects H_0 if and only if $X \geqslant k$. Is ϕ_k admissible as a test of H_0 against H_θ for every $\theta \in (0, 1)$? Is it uniformly most powerful for its size for testing H_0 against the composite hypothesis $H_1 : \Theta \in (0, 1)$? Is it admissible as a test of H_0 against H_1?

28J Stochastic Financial Models

(a) Let $(B_t : t \geqslant 0)$ be a Brownian motion and consider the process

$$Y_t = Y_0 e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t}$$

for $Y_0 > 0$ deterministic. For which values of μ is $(Y_t : t \geqslant 0)$ a supermartingale? For which values of μ is $(Y_t : t \geqslant 0)$ a martingale? For which values of μ is $(1/Y_t : t \geqslant 0)$ a martingale? Justify your answers.

(b) Assume that the riskless rates of return for Dollar investors and Euro investors are r_D and r_E respectively. Thus, 1 Dollar at time 0 in the bank account of a Dollar investor will grow to $e^{r_D t}$ Dollars at time t. For a Euro investor, the Dollar is a risky, tradable asset. Let $\mathbb{Q}_E$ be his equivalent martingale measure and assume that the EUR/USD exchange rate at time t, that is, the number of Euros that one Dollar will buy at time t, is given by

$$Y_t = Y_0 e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t},$$

where (B_t) is a Brownian motion under $\mathbb{Q}_E$. Determine μ as function of r_D and r_E. Verify that Y is a martingale if $r_D = r_E$.

(c) Let r_D, r_E be as in part (b). Let now $\mathbb{Q}_D$ be an equivalent martingale measure for a Dollar investor and assume that the EUR/USD exchange rate at time t is given by

$$Y_t = Y_0 e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t},$$

where now (B_t) is a Brownian motion under $\mathbb{Q}_D$. Determine μ as function of r_D and r_E. Given $r_D = r_E$, check, under $\mathbb{Q}_D$, that is Y is not a martingale but that $1/Y$ is a martingale.

(d) Assuming still that $r_D = r_E$, rederive the final conclusion of part (c), namely the martingale property of $1/Y$, directly from part (b).

29I Optimization and Control

Consider a stochastic controllable dynamical system P with action-space A and countable state-space S. Thus $P = (p_{xy}(a) : x, y \in S,\, a \in A)$ and $p_{xy}(a)$ denotes the transition probability from x to y when taking action a. Suppose that a cost $c(x,a)$ is incurred each time that action a is taken in state x, and that this cost is uniformly bounded. Write down the dynamic optimality equation for the problem of minimizing the expected long-run average cost.

State in terms of this equation a general result, which can be used to identify an optimal control and the minimal long-run average cost.

A particle moves randomly on the integers, taking steps of size 1. Suppose we can choose at each step a control parameter $u \in [\alpha, 1-\alpha]$, where $\alpha \in (0, 1/2)$ is fixed, which has the effect that the particle moves in the positive direction with probability u and in the negative direction with probability $1-u$. It is desired to maximize the long-run proportion of time π spent by the particle at 0. Show that there is a solution to the optimality equation for this example in which the relative cost function takes the form $\theta(x) = \mu\,|x|$, for some constant μ.

Determine an optimal control and show that the maximal long-run proportion of time spent at 0 is given by

$$\pi = \frac{1-2\alpha}{2\,(1-\alpha)}\,.$$

You may assume that it is valid to use an unbounded function θ in the optimality equation in this example.

30C Partial Differential Equations

(i) Define the concept of "fundamental solution" of a linear constant-coefficient partial differential operator and write down the fundamental solution for the operator $-\Delta$ on $\mathbb{R}^3$.

(ii) State and prove the mean value property for harmonic functions on $\mathbb{R}^3$.

(iii) Let $u \in C^2(\mathbb{R}^3)$ be a harmonic function which satisfies $u(p) \geqslant 0$ at every point p in an open set $\Omega \subset \mathbb{R}^3$. Show that if $B(z,r) \subset B(w,R) \subset \Omega$, then

$$u(w) \geqslant \left(\frac{r}{R}\right)^3 u\,(z)\,.$$

Assume that $B(x,4r) \subset \Omega$. Deduce, by choosing $R = 3r$ and w, z appropriately, that

$$\inf_{B(x,r)} u \geqslant 3^{-3} \sup_{B(x,r)} u\,.$$

[In (iii), $B(z,\rho) = \{x \in \mathbb{R}^3 : \|x - z\| < \rho\}$ is the ball of radius $\rho > 0$ centred at $z \in \mathbb{R}^3$.]

31C Integrable Systems

Describe the inverse scattering transform for the KdV equation, paying particular attention to the Lax representation and the evolution of the scattering data.

[*Hint: you may find it helpful to consider the operator*

$$A = 4\frac{d^3}{dx^3} - 3\Big(u\frac{d}{dx} + \frac{d}{dx}u\Big).]$$

32D Principles of Quantum Mechanics

Derive approximate expressions for the eigenvalues of a Hamiltonian $H + \lambda V$, working to second order in the parameter λ and assuming the eigenstates and eigenvalues of H are known and non-degenerate.

Let $\mathbf{J} = (J_1, J_2, J_3)$ be angular momentum operators with $|j\, m\rangle$ joint eigenstates of $\mathbf{J}^2$ and J_3. What are the possible values of the labels j and m and what are the corresponding eigenvalues of the operators?

A particle with spin j is trapped in space (its position and momentum can be ignored) but is subject to a magnetic field of the form $\mathbf{B} = (B_1, 0, B_3)$, resulting in a Hamiltonian $-\gamma(B_1 J_1 + B_3 J_3)$. Starting from the eigenstates and eigenvalues of this Hamiltonian when $B_1 = 0$, use perturbation theory to compute the leading order corrections to the energies when B_1 is non-zero but much smaller than B_3. Compare with the exact result.

[You may set $\hbar = 1$ and use $J_\pm|j\, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\, m{\pm}1\rangle$.]

33E Applications of Quantum Mechanics

Consider a large, essentially two-dimensional, rectangular sample of conductor of area A, and containing $2N$ electrons of charge $-e$. Suppose a magnetic field of strength B is applied perpendicularly to the sample. Write down the Landau Hamiltonian for one of the electrons assuming that the electron interacts just with the magnetic field.

[You may ignore the interaction of the electron spin with the magnetic field.]

Find the allowed energy levels of the electron.

Find the total energy of the $2N$ electrons at absolute zero temperature as a function of B, assuming that B is in the range

$$\frac{\pi\hbar N}{eA} \leqslant B \leqslant \frac{2\pi\hbar N}{eA}.$$

Comment on the values of the total energy when B takes the values at the two ends of this range.

34E Statistical Physics

Prove that energy fluctuations in a canonical distribution are given by

$$\left\langle (E - \langle E \rangle)^2 \right\rangle \;=\; k_B T^2 C_V$$

where T is the absolute temperature, $C_V = \frac{\partial \langle E \rangle}{\partial T}|_V$ is the heat capacity at constant volume, and k_B is Boltzmann's constant.

Prove the following relation in a similar manner:

$$\left\langle (E - \langle E \rangle)^3 \right\rangle \;=\; k_B^2 \left[T^4 \left.\frac{\partial C_V}{\partial T}\right|_V \;+\; 2T^3 C_V \right].$$

Show that, for an ideal gas of N monatomic molecules where $\langle E \rangle = \frac{3}{2} N k_B T$, these equations can be reduced to

$$\frac{1}{\langle E \rangle^2} \left\langle (E - \langle E \rangle)^2 \right\rangle = \frac{2}{3N} \quad \text{and} \quad \frac{1}{\langle E \rangle^3} \left\langle (E - \langle E \rangle)^3 \right\rangle = \frac{8}{9N^2}.$$

35E General Relativity

Let $x^a(\lambda)$ be a path P with tangent vector $T^a = \frac{d}{d\lambda}x^a(\lambda)$. For vectors $X^a(x(\lambda))$ and $Y^a(x(\lambda))$ defined on P let

$$\nabla_T X^a = \frac{d}{d\lambda}X^a + \Gamma^a{}_{bc}(x(\lambda))X^b T^c,$$

where $\Gamma^a{}_{bc}(x)$ is the metric connection for a metric $g_{ab}(x)$. $\nabla_T Y^a$ is defined similarly. Suppose P is geodesic and λ is an affine parameter. Explain why $\nabla_T T^a = 0$. Show that if $\nabla_T X^a = \nabla_T Y^a = 0$ then $g_{ab}(x(\lambda))X^a(x(\lambda))Y^b(x(\lambda))$ is constant along P.

If $x^a(\lambda, \mu)$ is a family of geodesics which depend on μ, let $S^a = \frac{\partial}{\partial\mu}x^a$ and define

$$\nabla_S X^a = \frac{\partial}{\partial\mu}X^a + \Gamma^a{}_{bc}(x(\lambda))X^b S^c.$$

Show that $\nabla_T S^a = \nabla_S T^a$ and obtain

$$\nabla_T{}^2 S^a \equiv \nabla_T(\nabla_T S^a) = R^a{}_{bcd}T^b T^c S^d.$$

What is the physical relevance of this equation in general relativity? Describe briefly how this is relevant for an observer moving under gravity.

[You may assume $[\nabla_T, \nabla_S]X^a = R^a{}_{bcd}X^b T^c S^d$.]

36A Fluid Dynamics II

Viscous fluid with dynamic viscosity μ flows with velocity $(u_x, u_y, u_z) \equiv (\mathbf{u}_H, u_z)$ (in cartesian coordinates x, y, z) in a shallow container with a free surface at $z = 0$. The base of the container is rigid, and is at $z = -h(x, y)$. A horizontal stress $\mathbf{S}(x, y)$ is applied at the free surface. Gravity may be neglected.

Using lubrication theory (conditions for the validity of which should be clearly stated), show that the horizontal volume flux $\mathbf{q}(x, y) \equiv \int_{-h}^{0} \mathbf{u}_H \, dz$ satisfies the equations

$$\nabla \cdot \mathbf{q} = 0 , \qquad \mu \mathbf{q} = -\frac{1}{3} h^3 \nabla p + \frac{1}{2} h^2 \mathbf{S} ,$$

where $p(x, y)$ is the pressure. Find also an expression for the surface velocity $\mathbf{u}_0(x, y) \equiv \mathbf{u}_H(x, y, 0)$ in terms of $\mathbf{S}$, $\mathbf{q}$ and h.

Now suppose that the container is cylindrical with boundary at $x^2 + y^2 = a^2$, where $a \gg h$, and that the surface stress is uniform and in the x-direction, so $\mathbf{S} = (S_0, 0)$ with S_0 constant. It can be assumed that the correct boundary condition to apply at $x^2 + y^2 = a^2$ is $\mathbf{q} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal.

Write $\mathbf{q} = \nabla \psi(x, y) \times \hat{\mathbf{z}}$, and show that ψ satisfies the equation

$$\nabla \cdot \left(\frac{1}{h^3} \nabla \psi \right) = - \frac{S_0}{2 \mu h^2} \frac{\partial h}{\partial y} .$$

Deduce that if $h = h_0$ (constant) then $\mathbf{q} = \mathbf{0}$. Find $\mathbf{u_0}$ in this case.

Now suppose that $h = h_0(1 + \epsilon y/a)$, where $\epsilon \ll 1$. Verify that to leading order in ϵ, $\psi = \epsilon C(x^2 + y^2 - a^2)$ for some constant C to be determined. Hence determine $\mathbf{u}_0$ up to and including terms of order ϵ.

[*Hint:* $\nabla \times (\mathbf{A} \times \hat{\mathbf{z}}) = \hat{\mathbf{z}} \cdot \nabla \mathbf{A} - \hat{\mathbf{z}} \nabla \cdot \mathbf{A}$ *for any vector field* $\mathbf{A}$.]

37B Waves

Show that, in one-dimensional flow of a perfect gas at constant entropy, the Riemann invariants $u \pm 2(c - c_0)/(\gamma - 1)$ are constant along characteristics $dx/dt = u \pm c$.

A perfect gas occupies a tube that lies parallel to the x-axis. The gas is initially at rest and is in $x > 0$. For times $t > 0$ a piston is pulled out of the gas so that its position at time t is

$$x = X(t) = -\frac{1}{2} f t^2,$$

where $f > 0$ is a constant. Sketch the characteristics of the resulting motion in the (x, t) plane and explain why no shock forms in the gas.

Calculate the pressure exerted by the gas on the piston for times $t > 0$, and show that at a finite time t_v a vacuum forms. What is the speed of the piston at $t = t_v$?

38C Numerical Analysis

The advection equation

$$u_t = u_x, \qquad x \in \mathbb{R}, \quad t \geqslant 0,$$

is solved by the leapfrog scheme

$$u_m^{n+1} = \mu \left(u_{m+1}^n - u_{m-1}^n \right) + u_m^{n-1},$$

where $n \geqslant 1$ and $\mu = \Delta t / \Delta x$ is the Courant number.

(a) Determine the local error of the method.

(b) Applying the Fourier technique, find the range of $\mu > 0$ for which the method is stable.

END OF PAPER