UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday 2 June 2008 9.00 to 12.00

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Define the continued fraction of a real number α.

Compute the continued fraction of $\sqrt{19}$.

2F Topics in Analysis

Let $P_0, P_1, P_2, \ldots$ be non-zero orthogonal polynomials on an interval $[a,b]$ such that the degree of P_j is equal to j for every $j = 0,1,2, \ldots,$ where the orthogonality is with respect to the inner product $< f,g > = \int_a^b fg$. If f is any continuous function on $[a,b]$ orthogonal to $P_0, P_1, \ldots, P_{n-1}$ and not identically zero, prove that f must have at least n distinct zeros in (a,b).

3G Geometry of Group Actions

Prove that an isometry of Euclidean space $\mathbb{R}^3$ is an affine transformation.

Deduce that a finite group of isometries of $\mathbb{R}^3$ has a common fixed point.

4G Coding and Cryptography

Define the entropy $H(X)$ of a random variable X that takes no more than N different values. What are the maximum and the minimum values for the entropy for a fixed value of N? Explain when the maximum and minimum are attained. You should prove any inequalities that you use.

5J Statistical Modelling

Consider the following Binomial generalized linear model for data $y_1, \ldots, y_n$, with logit link function. The data $y_1, \ldots, y_n$ are regarded as observed values of independent random variables $Y_1, \ldots, Y_n$, where

$$Y_i \sim \mathrm{Bin}(1, \mu_i), \quad \log \frac{\mu_i}{1-\mu_i} = \beta^\top x_i, \quad i = 1, \ldots, n,$$

where β is an unknown p-dimensional parameter, and where $x_1, \ldots, x_n$ are known p-dimensional explanatory variables. Write down the likelihood function for $y = (y_1, \ldots, y_n)$ under this model.

Show that the maximum likelihood estimate $\hat{\beta}$ satisfies an equation of the form $X^\top y = X^\top \hat{\mu}$, where X is the $p \times n$ matrix with rows $x_1^\top, \ldots, x_n^\top$, and where $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)$, with $\hat{\mu}_i$ a function of x_i and $\hat{\beta}$, which you should specify.

Define the deviance $D(y; \hat{\mu})$ and find an explicit expression for $D(y; \hat{\mu})$ in terms of y and $\hat{\mu}$ in the case of the model above.

6B Mathematical Biology

A gene product with concentration g is produced by a chemical S of concentration s, is autocatalysed and degrades linearly according to the kinetic equation

$$\frac{dg}{dt} = f(g, s) = s + k\,\frac{g^2}{1+g^2} - g,$$

where $k > 0$ is a constant.

First consider the case $s = 0$. Show that if $k > 2$ there are two positive steady states, and determine their stability. Sketch the reaction rate $f(g, 0)$.

Now consider $s > 0$. Show that there is a single steady state if s exceeds a critical value. If the system starts in the steady state $g = 0$ with $s = 0$ and then s is increased sufficiently before decreasing back to zero, show that a biochemical switch can be achieved to a state $g = g_2$, whose value you should determine.

7A Dynamical Systems

Sketch the phase plane of the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + x^2 - ky, \end{aligned}$$

(i) for $k = 0$ and (ii) for $k = 1/10$. Include in your sketches any trajectories that are the separatrices of a saddle point. In case (ii) shade the domain of stability of the origin.

8C Further Complex Methods

The function F is defined by

$$F(z) = \int_0^\infty \frac{t^{z-1}}{(t+1)^2} dt.$$

For which values of z does the integral converge?

Show that, for these values,

$$F(z) = \frac{\pi(1-z)}{\sin(\pi z)}.$$

9A Classical Dynamics

The action for a system with generalized coordinates $q_i(t)$ for a time interval $[t_1, t_2]$ is given by

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt,$$

where L is the Lagrangian. The end point values $q_i(t_1)$ and $q_i(t_2)$ are fixed.

Derive Lagrange's equations from the principle of least action by considering the variation of S for all possible paths.

Define the momentum p_i conjugate to q_i. Derive a condition for p_i to be a constant of the motion.

A symmetric top moves under the action of a potential $V(\theta)$. The Lagrangian is given by

$$L = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - V,$$

where the generalized coordinates are the Euler angles (θ, ϕ, ψ) and the principal moments of inertia are I_1 and I_3.

Show that $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$ is a constant of the motion and give expressions for two others. Show further that it is possible for the top to move with both θ and $\dot{\phi}$ constant provided these satisfy the condition

$$I_1\dot{\phi}^2\sin\theta\cos\theta - I_3\omega_3\dot{\phi}\sin\theta = \frac{dV}{d\theta}.$$

10E Cosmology

The number density of particles of mass m at equilibrium in the early universe is given by the integral

$$n = \frac{4\pi g_{\mathrm{s}}}{h^3} \int_0^\infty \frac{p^2 dp}{\exp[(E(p)-\mu)/kT] \mp 1}, \qquad \begin{cases} - & \text{bosons}\,, \\ + & \text{fermions}, \end{cases}$$

where $E(p) = c\sqrt{p^2 + m^2c^2}$, μ is the chemical potential, and g_{s} is the spin degeneracy. Assuming that the particles remain in equilibrium when they become non-relativistic ($kT, \mu \ll mc^2$), show that the number density can be expressed as

$$n = g_{\mathrm{s}} \left(\frac{2\pi m kT}{h^2} \right)^{3/2} e^{(\mu - mc^2)/kT}.$$

[*Hint: Recall that* $\int_0^\infty dx\, e^{-\sigma^2 x^2} = \sqrt{\pi}/(2\sigma)$, $\quad (\sigma > 0)$.]

At around $t = 100$ seconds, deuterium D forms through the nuclear fusion of nonrelativistic protons p and neutrons n via the interaction $p + n \leftrightarrow D$. In equilibrium, what is the relationship between the chemical potentials of the three species? Show that the ratio of their number densities can be expressed as

$$\frac{n_D}{n_n n_p} \approx \left(\frac{\pi m_p kT}{h^2} \right)^{-3/2} e^{B_D/kT},$$

where the deuterium binding energy is $B_D = (m_n + m_p - m_D)\, c^2$ and you may take $g_D = 4$. Now consider the fractional densities $X_a = n_a/n_B$, where n_B is the baryon density of the universe, to re-express the ratio above in the form $X_D/(X_n X_p)$, which incorporates the baryon-to-photon ratio η of the universe.

[You may assume that the photon density is $n_\gamma = (16\pi\zeta(3)/(hc)^3)(kT)^3$.]

Why does deuterium form only at temperatures much lower than that given by $kT \approx B_D$?

SECTION II

11G Geometry of Group Actions

What is meant by an *inversion* in a circle in $\mathbb{C} \cup \{\infty\}$? Show that a composition of two inversions is a Möbius transformation.

Hence, or otherwise, show that if C^+ and C^- are two disjoint circles in $\mathbb{C}$, then the composition of the inversions in C^+ and C^- has two fixed points.

12G Coding and Cryptography

State Shannon's Noisy Coding Theorem for a binary symmetric channel.

Define the *mutual information* of two discrete random variables X and Y. Prove that the mutual information is symmetric and non-negative. Define also the *information capacity* of a channel.

A channel transmits numbers chosen from the alphabet $\mathcal{A} = \{0, 1, 2\}$ and has transition matrix

$$\begin{pmatrix} 1-2\beta & \beta & \beta \\ \beta & 1-2\beta & \beta \\ \beta & \beta & 1-2\beta \end{pmatrix}$$

for a number β with $0 \leqslant \beta \leqslant \frac{1}{3}$. Calculate the information capacity of the channel.

13J Statistical Modelling

Consider performing a two-way analysis of variance (ANOVA) on the following data:

```
> Y[,,1]            Y[,,2]                  Y[,,3]
      [,1] [,2]          [,1]    [,2]           [,1]   [,2]
 [1,] 2.72 6.66    [1,] -5.780  1.7200    [1,] -2.2900 0.158
 [2,] 4.88 5.98    [2,] -4.600  1.9800    [2,] -3.1000 1.190
 [3,] 3.49 8.81    [3,] -1.460  2.1500    [3,] -2.6300 1.190
 [4,] 2.03 6.26    [4,] -1.780  0.7090    [4,] -0.2400 1.470
 [5,] 2.39 8.50    [5,] -2.610 -0.5120    [5,]  0.0637 2.110
   .    .    .       .      .       .          .      .      .
   .    .    .       .      .       .          .      .      .
   .    .    .       .      .       .          .      .      .
```

Explain and interpret the `R` commands and (slightly abbreviated) output below. In particular, you should describe the model being fitted, and comment on the hypothesis tests which are performed under the `summary` and `anova` commands.

```
> K <- dim(Y)[1]
> I <- dim(Y)[2]
> J <- dim(Y)[3]
> c(I,J,K)
[1]  2  3 10
> y <- as.vector(Y)
> a <- gl(I, K, length(y))
> b <- gl(J, K * I, length(y))
> fit1 <- lm(y ~ a + b)
> summary(fit1)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   3.7673     0.3032   12.43  < 2e-16 ***
a2            3.4542     0.3032   11.39 3.27e-16 ***
b2           -6.3215     0.3713  -17.03  < 2e-16 ***
b3           -5.8268     0.3713  -15.69  < 2e-16 ***
> anova(fit1)
```

```
Response: y
          Df Sum Sq Mean Sq F value    Pr(>F)
a          1 178.98  178.98  129.83 3.272e-16 ***
b          2 494.39  247.19  179.31 < 2.2e-16 ***
Residuals 56  77.20    1.38
```

The following `R` code fits a similar model. Briefly explain the difference between this model and the one above. Based on the output of the `anova` call below, say whether you prefer this model over the one above, and explain your preference.

```
> fit2 <- lm(y ~ a * b)
> anova(fit2)
Response: y
          Df Sum Sq Mean Sq  F value    Pr(>F)
a          1 178.98  178.98 125.6367 1.033e-15 ***
b          2 494.39  247.19 173.5241 < 2.2e-16 ***
a:b        2   0.27    0.14   0.0963    0.9084
Residuals 54  76.93    1.42
```

Finally, explain what is being calculated in the code below and give the value that would be obtained by the final line of code.

```
> n <- I * J * K
> p <- length(coef(fit2))
> p0 <- length(coef(fit1))
> PY <- fitted(fit2)
> P0Y <- fitted(fit1)
> ((n - p)/(p - p0)) * sum((PY - P0Y)^2)/sum((y - PY)^2)
```

14C Further Complex Methods

Show that under the change of variable $z = \sin^2 x$ the equation

$$\frac{d^2w}{dx^2} + n^2 w = 0$$

becomes

$$\frac{d^2w}{dz^2} + \frac{2z-1}{2z(z-1)}\frac{dw}{dz} - \frac{n^2}{4(z-1)z}w = 0.$$

Show that this is a Papperitz equation and that the corresponding P-function is

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \frac{1}{2}n & 0 & z \\ \frac{1}{2} & -\frac{1}{2}n & \frac{1}{2} & \end{matrix}\right\}.$$

Deduce that $F(\frac{1}{2}n, -\frac{1}{2}n; \frac{1}{2}; \sin^2 x) = \cos nx$.

15E Cosmology

(i) A homogeneous and isotropic universe has mass density $\rho(t)$ and scale factor $a(t)$. Show how the conservation of total energy (kinetic plus gravitational potential) when applied to a test particle on the edge of a spherical region in this universe can be used to obtain the Friedmann equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2},$$

where k is a constant. State clearly any assumptions you have made.

(ii) Assume that the universe is flat ($k = 0$) and filled with two major components: pressure-free matter ($P_{\mathrm{M}} = 0$) and dark energy with equation of state $P_\Lambda = -\rho_\Lambda c^2$ where their mass densities today ($t = t_0$) are given respectively by $\rho_{\mathrm{M}0}$ and $\rho_{\Lambda 0}$. Assuming that each component independently satisfies the fluid conservation equation, $\dot{\rho} = -3H(\rho + P/c^2)$, show that the total mass density can be expressed as

$$\rho(t) = \frac{\rho_{\mathrm{M}0}}{a^3} + \rho_{\Lambda 0},$$

where we have set $a(t_0) = 1$.

Hence, solve the Friedmann equation and show that the scale factor can be expressed in the form

$$a(t) = \alpha(\sinh \beta t)^{2/3},$$

where α and β are constants which you should specify in terms of $\rho_{\mathrm{M}0}$, $\rho_{\Lambda 0}$ and t_0.

[*Hint: try the substitution* $b = a^{3/2}$.]

Show that the scale factor $a(t)$ has the expected behaviour for a matter-dominated universe at early times ($t \to 0$) and that the universe accelerates at late times ($t \to \infty$).

16G Logic and Set Theory

What is a well-ordered set? Show that given any two well-ordered sets there is a unique order isomorphism between one and an initial segment of the other.

Show that for any ordinal α and for any non-zero ordinal β there are unique ordinals γ and δ with $\alpha = \beta.\gamma + \delta$ and $\delta < \beta$.

Show that a non-zero ordinal λ is a limit ordinal if and only if $\lambda = \omega.\gamma$ for some non-zero ordinal γ .

[You may assume standard properties of ordinal addition, multiplication and subtraction.]

17F Graph Theory

State a result of Euler concerning the number of vertices, edges and faces of a connected plane graph. Deduce that if G is a planar graph then $\delta(G) \leqslant 5$. Show that if G is a planar graph then $\chi(G) \leqslant 5$.

Are the following statements true or false? Justify your answers.

[You may quote standard facts about planar and non-planar graphs, provided that they are clearly stated.]

(i) If G is a graph with $\chi(G) \leqslant 4$ then G is planar.

(ii) If G is a connected graph with average degree at most 2.01 then G is planar.

(iii) If G is a connected graph with average degree at most 2 then G is planar.

18H Galois Theory

Find the Galois group of the polynomial $f(x) = x^4 + x^3 + 1$ over

(i) the finite field $\mathbf{F}_2$, (ii) the finite field $\mathbf{F}_3$,

(iii) the finite field $\mathbf{F}_4$, (iv) the field $\mathbf{Q}$ of rational numbers.

[Results from the course which you use should be stated precisely.]

19G Representation Theory

For a complex representation V of a finite group G, define the action of G on the dual representation V^*. If α denotes the character of V, compute the character β of V^*.

[Your formula should express $\beta(g)$ just in terms of the character α.]

Using your formula, how can you tell from the character whether a given representation is self-dual, that is, isomorphic to the dual representation?

Let V be an irreducible representation of G. Show that the trivial representation occurs as a summand of $V \otimes V$ with multiplicity either 0 or 1. Show that it occurs once if and only if V is self-dual.

For a self-dual irreducible representation V, show that V either has a nondegenerate G-invariant symmetric bilinear form or a nondegenerate G-invariant alternating bilinear form, but not both.

If V is an irreducible self-dual representation of odd dimension n, show that the corresponding homomorphism $G \to GL(n, \mathbf{C})$ is conjugate to a homomorphism into the orthogonal group $O(n, \mathbf{C})$. Here $O(n, \mathbf{C})$ means the subgroup of $GL(n, \mathbf{C})$ that preserves a nondegenerate symmetric bilinear form on $\mathbf{C}^n$.

20G Number Fields

(a) Define the ideal class group of an algebraic number field K. State a result involving the discriminant of K that implies that the ideal class group is finite.

(b) Put $K = \mathbb{Q}(\omega)$, where $\omega = \frac{1}{2}(1 + \sqrt{-23})$, and let $\mathcal{O}_K$ be the ring of integers of K. Show that $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$. Factorise the ideals [2] and [3] in $\mathcal{O}_K$ into prime ideals. Verify that the equation of ideals

$$[2, \omega][3, \omega] = [\omega]$$

holds. Hence prove that K has class number 3.

21F Algebraic Topology

(i) State the van Kampen theorem.

(ii) Calculate the fundamental group of the wedge $S^2 \vee S^1$.

(iii) Let $X = \mathbb{R}^3 \setminus A$ where A is a circle. Calculate the fundamental group of X.

22F Linear Analysis

Suppose p and q are real numbers with $p^{-1} + q^{-1} = 1$ and p, $q > 1$. Show, quoting any results on convexity that you need, that

$$a^{1/p}\, b^{1/q} \leqslant \frac{a}{p} + \frac{b}{q}$$

for all real positive a and b.

Define the space l^p and show that it is a complete normed vector space.

23H Riemann Surfaces

Define the terms *Riemann surface*, *holomorphic map* between Riemann surfaces and *biholomorphic map*.

Show, without using the notion of degree, that a non-constant holomorphic map between compact connected Riemann surfaces must be surjective.

Let ϕ be a biholomorphic map of the punctured unit disc $\Delta^* = \{0 < |z| < 1\} \subset \mathbb{C}$ onto itself. Show that ϕ extends to a biholomorphic map of the open unit disc Δ to itself such that $\phi(0) = 0$.

Suppose that $f : R \to S$ is a continuous holomorphic map between Riemann surfaces and f is holomorphic on $R \setminus \{p\}$, where p is a point in R. Show that f is then holomorphic on all of R.

[The Open Mapping Theorem may be used without proof if clearly stated.]

24H Differential Geometry

Let $n \geqslant 1$ be an integer, and let $M(n)$ denote the set of $n \times n$ real-valued matrices. We make $M(n)$ into an n^2-dimensional smooth manifold via the obvious identification $M(n) = \mathbb{R}^{n^2}$.

(a) Let $GL(n)$ denote the subset

$$GL(n) = \{A \in M(n) : A^{-1} \text{ exists}\}.$$

Show that $GL(n)$ is a submanifold of $M(n)$. What is $\dim GL(n)$?

(b) Now let $SL(n) \subset GL(n)$ denote the subset

$$SL(n) = \{A \in GL(n) : \det A = 1\}.$$

Show that for $A \in GL(n)$,

$$(d \det)_A B = \operatorname{tr}(A^{-1}B) \det A.$$

Show that $SL(n)$ is a submanifold of $GL(n)$. What is the dimension of $SL(n)$?

(c) Now consider the set $X = M(n) \setminus GL(n)$. For what values of $n \geqslant 1$ is X a submanifold of $M(n)$?

25J Probability and Measure

State the Dominated Convergence Theorem.

Hence or otherwise prove Kronecker's Lemma: if (a_j) is a sequence of non-negative reals such that

$$\sum_{j=1}^{\infty} \frac{a_j}{j} < \infty,$$

then

$$n^{-1} \sum_{j=1}^{n} a_j \to 0 \quad (n \to \infty).$$

Let $\xi_1, \xi_2, \ldots$ be independent $N(0,1)$ random variables and set $S_n = \xi_1 + \ldots + \xi_n$. Let $\mathcal{F}_0$ be the collection of all finite unions of intervals of the form (a,b), where a and b are rational, together with the whole line $\mathbb{R}$. Prove that with probability 1 the limit

$$m(B) \equiv \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} I_B(S_j)$$

exists for all $B \in \mathcal{F}_0$, and identify it. Is it possible to extend m defined on $\mathcal{F}_0$ to a measure on the Borel σ-algebra of $\mathbb{R}$? Justify your answer.

26I Applied Probability

Let $(X_t, t \geqslant 0)$ be an irreducible continuous-time Markov chain with initial probability distribution π and Q-matrix Q (for short: a (π, Q) CTMC), on a finite state space I.

(i) Define the terms *reversible* CTMC and *detailed balance equations* (DBEs) and explain, without proof, the relation between them.

(ii) Prove that any solution of the DBEs is an equilibrium distribution (ED) for (X_t).

Let $(Y_n, n = 0, 1, \ldots)$ be an irreducible discrete-time Markov chain with initial probability distribution $\widehat{\pi}$ and transition probability matrix $\widehat{P}$ (for short: a $(\widehat{\pi}, \widehat{P})$ DTMC), on the state space I.

(iii) Repeat the two definitions from (i) in the context of the DTMC (Y_n). State also in this context the relation between them, and prove a statement analogous to (ii).

(iv) What does it mean to say that (Y_n) is the *jump chain* for (X_t)? State and prove a relation between the ED π for the CTMC (X_t) and the ED $\widehat{\pi}$ for its jump chain (Y_n).

(v) Prove that (X_t) is reversible (in equilibrium) if and only if its jump chain (Y_n) is reversible (in equilibrium).

(vi) Consider now a continuous time random walk on a graph. More precisely, consider a CTMC (X_t) on an undirected graph, where some pairs of states $i, j \in I$ are joined by one or more non-oriented 'links' $e_{ij}(1), \ldots, e_{ij}(m_{ij})$. Here $m_{ij} = m_{ji}$ is the number of links between i and j. Assume that the jump rate q_{ij} is proportional to m_{ij}. Can the chain (X_t) be reversible? Identify the corresponding jump chain (Y_n) (which determines a discrete-time random walk on the graph) and comment on its reversibility.

27I Principles of Statistics

An angler starts fishing at time 0. Fish bite in a Poisson Process of rate Λ per hour, so that, if $\Lambda = \lambda$, the number N_t of fish he catches in the first t hours has the Poisson distribution $\mathcal{P}(\lambda t)$, while T_n, the time in hours until his nth bite, has the Gamma distribution $\Gamma(n, \lambda)$, with density function

$$p(t \mid \lambda) = \frac{\lambda^n}{(n-1)!} \, t^{n-1} \, e^{-\lambda t} \quad (t > 0) \, .$$

Bystander B_1 plans to watch for 3 hours, and to record the number N_3 of fish caught. Bystander B_2 plans to observe until the 10th bite, and to record T_{10} , the number of hours until this occurs.

For B_1 , show that $\widetilde{\Lambda}_1 := N_3/3$ is an unbiased estimator of Λ whose variance function achieves the Cramér–Rao lower bound.

Find an unbiased estimator of Λ for B_2, of the form $\widetilde{\Lambda}_2 = k/T_{10}$. Does it achieve the Cramér–Rao lower bound? Is it minimum-variance-unbiased? Justify your answers.

In fact, the 10th fish bites after exactly 3 hours. For each of B_1 and B_2, write down the likelihood function for Λ based their observations. What does the *Likelihood Principle* have to say about the inferences to be drawn by B_1 and B_2, and why? Compute the estimates $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ produced by applying $\widetilde{\Lambda}_1$ and $\widetilde{\Lambda}_2$ to the observed data. Does the method of minimum-variance-unbiased estimation respect the Likelihood Principle?

28J Stochastic Financial Models

(a) In the context of the Black–Scholes formula, let S_0 be the time-0 spot price, K be the strike price, T be the time to maturity, and let σ be the volatility. Assume that the interest rate r is constant and assume absence of dividends. Write $\mathrm{EC}(S_0, K, \sigma, r, T)$ for the time-0 price of a standard European call. The Black–Scholes formula can be written in the following form

$$\mathrm{EC}(S_0, K, \sigma, r, T) = S_0 \Phi(d_1) - e^{-rT} K \Phi(d_2).$$

State how the quantities d_1 and d_2 depend on S_0, K, σ, r and T.

Assume that you sell this option at time 0. What is your replicating portfolio at time 0?

[No proofs are required.]

(b) Compute the limit of $\mathrm{EC}(S_0, K, \sigma, r, T)$ as $\sigma \to \infty$. Construct an explicit arbitrage under the assumption that European calls are traded above this limiting price.

(c) Compute the limit of $\mathrm{EC}(S_0, K, \sigma, r, T)$ as $\sigma \to 0$. Construct an explicit arbitrage under the assumption that European calls are traded below this limiting price.

(d) Express in terms of S_0, d_1 and T the derivative

$$\frac{\partial}{\partial \sigma} \mathrm{EC}(S_0, K, \sigma, r, T).$$

[*Hint: you may find it useful to express* $\frac{\partial}{\partial \sigma} d_1$ *in terms of* $\frac{\partial}{\partial \sigma} d_2$.]

[You may use without proof the formula $S_0 \Phi'(d_1) - e^{-rT} K \Phi'(d_2) = 0$.]

(e) Say what is meant by implied volatility and explain why the previous results make it well-defined.

29C Partial Differential Equations

(i) State the local existence theorem for the first order quasi-linear partial differential equation

$$\sum_{j=1}^{n} a_j(x,u)\,\frac{\partial u}{\partial x_j} = b(x,u),$$

which is to be solved for a real-valued function with data specified on a hypersurface S. Include a definition of "non-characteristic" in your answer.

(ii) Consider the linear constant-coefficient case (that is, when all the functions $a_1,\ldots,a_n$ are real constants and $b(x,u) = cx + d$ for some $c = (c_1,\ldots,c_n)$ with $c_1,\ldots,c_n$ real and d real) and with the hypersurface S taken to be the hyperplane $\mathbf{x}\cdot\mathbf{n} = 0$. Explain carefully the relevance of the non-characteristic condition in obtaining a solution via the method of characteristics.

(iii) Solve the equation

$$\frac{\partial u}{\partial y} + u\,\frac{\partial u}{\partial x} = 0,$$

with initial data $u(0,y) = -y$ prescribed on $x = 0$, for a real-valued function $u(x,y)$. Describe the domain on which your solution is C^1 and comment on this in relation to the theorem stated in (i).

30A Asymptotic Methods

Obtain an expression for the nth term of an asymptotic expansion, valid as $\lambda \to \infty$, for the integral

$$I(\lambda) = \int_0^1 t^{2\alpha} e^{-\lambda(t^2+t^3)}\,dt \qquad (\alpha > -1/2).$$

Estimate the value of n for the term of least magnitude.

Obtain the first two terms of an asymptotic expansion, valid as $\lambda \to \infty$, for the integral

$$J(\lambda) = \int_0^1 t^{2\alpha} e^{-\lambda(t^2-t^3)}\,dt \qquad (-1/2 < \alpha < 0)\,.$$

[*Hint:*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt\,.]$$

[Stirling's formula may be quoted.]

31C Integrable Systems

Define an integrable system in the context of Hamiltonian mechanics with a finite number of degrees of freedom and state the Arnold–Liouville theorem.

Consider a six-dimensional phase space with its canonical coordinates (p_j, q_j), $j = 1, 2, 3$, and the Hamiltonian

$$\frac{1}{2}\sum_{j=1}^{3} {p_j}^2 + F(r),$$

where $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$ and where F is an arbitrary function. Show that both $M_1 = q_2 p_3 - q_3 p_2$ and $M_2 = q_3 p_1 - q_1 p_3$ are first integrals.

State the Jacobi identity and deduce that the Poisson bracket

$$M_3 = \{M_1, M_2\}$$

is also a first integral. Construct a suitable expression out of M_1, M_2, M_3 to demonstrate that the system admits three first integrals in involution and thus satisfies the hypothesis of the Arnold–Liouville theorem.

32D Principles of Quantum Mechanics

(a) If A and B are operators which each commute with their commutator $[A,B]$, show that $[A,e^B]=[A,B]e^B$. By considering

$$F(\lambda)=e^{\lambda A}e^{\lambda B}e^{-\lambda(A+B)}$$

and differentiating with respect to the parameter λ, show also that

$$e^A e^B \;=\; Ce^{A+B} \;=\; e^{A+B}C$$

where $C=e^{\frac{1}{2}[A,B]}$.

(b) Consider a one-dimensional quantum system with position $\hat{x}$ and momentum $\hat{p}$. Write down a formula for the operator $U(\alpha)$ corresponding to translation through α, calculate $[\hat{x},U(\alpha)]$, and show that your answer is consistent with the assumption that position eigenstates obey $|x+\alpha\rangle = U(\alpha)|x\rangle$. Given this assumption, express the wavefunction for $U(\alpha)|\psi\rangle$ in terms of the wavefunction $\psi(x)$ for $|\psi\rangle$.

Now suppose the one-dimensional system is a harmonic oscillator of mass m and frequency ω. Show that

$$\psi_0(x-\alpha) \;=\; e^{-m\omega\alpha^2/4\hbar}\sum_{n=0}^{\infty}\left(\frac{m\omega}{2\hbar}\right)^{n/2}\frac{\alpha^n}{\sqrt{n!}}\,\psi_n(x),$$

where $\psi_n(x)$ are normalised wavefunctions with energies $E_n=\hbar\omega(n+\frac{1}{2})$.

[Standard results for constructing normalised energy eigenstates in terms of annihilation and creation operators

$$a=\left(\frac{m\omega}{2\hbar}\right)^{1/2}\!\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right),\qquad a^\dagger=\left(\frac{m\omega}{2\hbar}\right)^{1/2}\!\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)$$

may be quoted without proof.]

33E Applications of Quantum Mechanics

A beam of particles each of mass m and energy $\hbar^2 k^2/(2m)$ scatters off an axisymmetric potential V. In the first Born approximation the scattering amplitude is

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \int e^{-i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{x}'} V(\mathbf{x}')\, d^3x', \qquad (*)$$

where $\mathbf{k}_0 = (0,0,k)$ is the wave vector of the incident particles and $\mathbf{k} = (k\sin\theta, 0, k\cos\theta)$ is the wave vector of the outgoing particles at scattering angle θ (and $\phi = 0$). Let $\mathbf{q} = \mathbf{k} - \mathbf{k}_0$ and $q = |\mathbf{q}|$. Show that when the scattering potential V is spherically symmetric the expression $(*)$ simplifies to

$$f(\theta) = -\frac{2m}{\hbar^2 q} \int_0^\infty r' V(r') \sin(qr')\, dr',$$

and find the relation between q and θ.

Calculate this scattering amplitude for the potential $V(r) = V_0 e^{-r}$ where V_0 is a constant, and show that at high energies the particles emerge predominantly in a narrow cone around the forward beam direction. Estimate the angular width of the cone.

34D Electrodynamics

Frame $\mathcal{S}'$ is moving with uniform speed v in the x-direction relative to a laboratory frame $\mathcal{S}$. The components of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in the two frames are related by the Lorentz transformation

$$E'_x = E_x, \quad E'_y = \gamma(E_y - vB_z), \quad E'_z = \gamma(E_z + vB_y),$$

$$B'_x = B_x, \quad B'_y = \gamma(B_y + vE_z), \quad B'_z = \gamma(B_z - vE_y),$$

where $\gamma = 1/\sqrt{1-v^2}$ and units are chosen so that $c = 1$. How do the components of the spatial vector $\mathbf{F} = \mathbf{E} + i\mathbf{B}$ (where $i = \sqrt{-1}$) transform?

Show that $\mathbf{F}'$ is obtained from $\mathbf{F}$ by a rotation through θ about a spatial axis $\mathbf{n}$, where $\mathbf{n}$ and θ should be determined. Hence, or otherwise, show that there are precisely two independent scalars associated with $\mathbf{F}$ which are preserved by the Lorentz transformation, and obtain them.

[*Hint: since* $|v| < 1$ *there exists a unique real* ψ *such that* $v = \tanh\psi$.]

35E General Relativity

For the metric

$$ds^2 = \frac{1}{r^2}\left(-dt^2 + dr^2\right), \qquad r \geqslant 0,$$

obtain the geodesic equations of motion. For a massive particle show that

$$\left(\frac{dr}{dt}\right)^2 = 1 - \frac{1}{k^2 r^2},$$

for some constant k. Show that the particle moves on trajectories

$$r^2 - t^2 = \frac{1}{k^2}, \qquad kr = \sec\tau, \quad kt = \tan\tau,$$

where τ is the proper time, if the origins of t, τ are chosen appropriately.

36A Fluid Dynamics II

Derive the relation between the stress tensor σ_{ij} and the rate-of-strain tensor e_{ij} in an incompressible Newtonian fluid, using the result that there is a linear dependence between the components of σ_{ij} and those of e_{ij} that is the same in all frames. Write down the boundary conditions that hold at an interface between two viscous fluids.

Viscous fluid is contained in a channel between the rigid planes $y = -a$ and $y = a$. The fluid in $y < 0$ has dynamic viscosity μ_-, while that in $y > 0$ has dynamic viscosity μ_+. Gravity may be neglected. The fluids move through the channel in the x-direction under the influence of a pressure gradient applied at the ends of the channel. It may be assumed that the velocity has no z-components, and all quantities are independent of z.

Find a steady solution of the Navier–Stokes equation in which the interface between the two fluids remains at $y = 0$, the fluid velocity is everywhere independent of x, and the pressure gradient is uniform. Use it to calculate the following:

(a) the viscous tangential stress at $y = -a$ and at $y = a$; and

(b) the ratio of the volume fluxes of the two different fluids.

Comment on the limits of each of the results in (a) and (b) as $\mu_+/\mu_- \to 1$, and as $\mu_+/\mu_- \to \infty$.

37B Waves

Show that in an acoustic plane wave the velocity and perturbation pressure are everywhere proportional and find the constant of proportionality.

Gas occupies a tube lying parallel to the x-axis. In the regions $x<0$ and $x>L$ the gas has uniform density ρ_0 and sound speed c_0 . For $0<x<L$ the gas is cooled so that it has uniform density ρ_1 and sound speed c_1 . A harmonic plane wave with frequency ω is incident from $x=-\infty$. Show that the amplitude of the wave transmitted into $x>L$ relative to that of the incident wave is

$$|T| \;=\; \left[\cos^2 k_1 L + \frac{1}{4}\left(\lambda+\lambda^{-1}\right)^2 \sin^2 k_1 L\right]^{-1/2} ,$$

where $\lambda=\rho_1 c_1/\rho_0 c_0$ and $k_1=\omega/c_1$.

What are the implications of this result if $\lambda \gg 1$?

38C Numerical Analysis

The Poisson equation $\nabla^2 u = f$ in the unit square $\Omega=[0,1]\times[0,1]$, with zero boundary conditions on $\partial\Omega$, is discretized with the nine-point formula

$$\begin{aligned}\frac{10}{3}\,u_{m,n} - \frac{2}{3}\,&(u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1})\\ &- \frac{1}{6}\,(u_{m+1,n+1}+u_{m+1,n-1}+u_{m-1,n+1}+u_{m-1,n-1}) = -h^2 f_{m,n},\end{aligned}$$

where $1\leqslant m,n\leqslant M$, $u_{m,n}\approx u(mh,nh)$, and (mh,nh) are grid points.

(a) Prove that, for any ordering of the grid points, the method can be written as $A\mathbf{u}=\mathbf{b}$ with a symmetric positive-definite matrix A.

(b) Describe the Jacobi method for solving a linear system of equations, and prove that it converges for the above system.

[You may quote without proof the corollary of the Householder–John theorem regarding convergence of the Jacobi method.]

END OF PAPER