UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 6 June 2008 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B,*** *…,* ***H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Describe (without proof) what it means to put an $n \times n$ matrix of complex numbers into Jordan normal form. Explain (without proof) the sense in which the Jordan normal form is unique.

Put the following matrix in Jordan normal form:

$$\begin{pmatrix} -7 & 3 & -5 \\ 7 & -1 & 5 \\ 17 & -6 & 12 \end{pmatrix}.$$

2G Groups, Rings and Modules

Let $n \geq 2$ be an integer. Show that the polynomial $(X^n - 1)/(X - 1)$ is irreducible over $\mathbb{Z}$ if and only if n is prime.

[*You may use Eisenstein's criterion without proof.*]

3F Analysis II

Let X be the vector space of all continuous real-valued functions on the unit interval $[0, 1]$. Show that the functions

$$\|f\|_1 = \int_0^1 |f(t)|\, dt \qquad \text{and} \qquad \|f\|_\infty = \sup\{|f(t)| : 0 \leqslant t \leqslant 1\}$$

both define norms on X.

Consider the sequence (f_n) defined by $f_n(t) = nt^n(1-t)$. Does (f_n) converge in the norm $\| - \|_1$? Does it converge in the norm $\| - \|_\infty$? Justify your answers.

4E Complex Analysis

Suppose that f and g are two functions which are analytic on the whole complex plane $\mathbb{C}$. Suppose that there is a sequence of distinct points $z_1, z_2, \ldots$ with $|z_i| \leqslant 1$ such that $f(z_i) = g(z_i)$. Show that $f(z) = g(z)$ for all $z \in \mathbb{C}$. [*You may assume any results on Taylor expansions you need, provided they are clearly stated.*]

What happens if the assumption that $|z_i| \leqslant 1$ is dropped?

5A Methods

Find the half-range Fourier cosine series for $f(x) = x^2$, $0 < x < 1$. Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \,.$$

6A Quantum Mechanics

What is meant by a stationary state? What form does the wavefunction take in such a state? A particle has wavefunction $\psi(x,t)$, such that

$$\psi(x,0) = \sqrt{\frac{1}{2}}\,(\chi_1(x) + \chi_2(x)) \ ,$$

where χ_1 and χ_2 are normalised eigenstates of the Hamiltonian with energies E_1 and E_2. Write down $\psi(x,t)$ at time t. Show that the expectation value of A at time t is

$$\langle A\rangle_\psi = \frac{1}{2}\int_{-\infty}^{\infty}\left(\chi_1^*\hat{A}\chi_1 + \chi_2^*\hat{A}\chi_2\right)dx + Re\left(e^{i(E_1-E_2)t/\hbar}\int_{-\infty}^{\infty}\chi_1^*\hat{A}\chi_2\,dx\right) \,.$$

7B Electromagnetism

The energy stored in a static electric field $\mathbf{E}$ is

$$U = \frac{1}{2}\int \rho\phi\, dV \ ,$$

where ϕ is the associated electric potential, $\mathbf{E} = -\nabla\phi$, and ρ is the volume charge density.

(i) Assuming that the energy is calculated over all space and that $\mathbf{E}$ vanishes at infinity, show that the energy can be written as

$$U = \frac{\epsilon_0}{2}\int |\mathbf{E}|^2 dV \ .$$

(ii) Find the electric field produced by a spherical shell with total charge Q and radius R, assuming it to vanish inside the shell. Find the energy stored in the electric field.

8D Numerical Analysis

Show that the Chebyshev polynomials, $T_n(x) = \cos(n \cos^{-1} x)\,, n = 0, 1, 2, \ldots$ obey the orthogonality relation

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2}\delta_{n,m}(1+\delta_{n,0}).$$

State briefly how an optimal choice of the parameters a_k, x_k, $k = 1, 2 \ldots n$ is made in the Gaussian quadrature formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \sim \sum_{k=1}^{n} a_k f(x_k).$$

Find these parameters for the case $n = 3$.

9H Markov Chains

A Markov chain on the state–space $I = \{1, 2, 3, 4, 5, 6, 7\}$ has transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/4 & 0 & 1/4 & 0 & 0 \\ 1/3 & 0 & 1/2 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

Classify the chain into its communicating classes, deciding for each what the period is, and whether the class is recurrent.

For each $i, j \in I$ say whether the limit $\lim_{n\to\infty} p_{ij}^{(n)}$ exists, and evaluate the limit when it does exist.

SECTION II

10E Linear Algebra

What is meant by a Hermitian matrix? Show that if A is Hermitian then all its eigenvalues are real and that there is an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A.

A Hermitian matrix is said to be *positive definite* if $\langle Ax, x\rangle > 0$ for all $x \neq 0$. We write $A > 0$ in this case. Show that A is positive definite if, and only if, all of its eigenvalues are positive. Show that if $A > 0$ then A has a unique positive definite square root $\sqrt{A}$.

Let A, B be two positive definite Hermitian matrices with $A - B > 0$. Writing $C = \sqrt{A}$ and $X = \sqrt{A} - \sqrt{B}$, show that $CX + XC > 0$. By considering eigenvalues of X, or otherwise, show that $X > 0$.

11G Groups, Rings and Modules

Let R be a ring and M an R-module. What does it mean to say that M is a free R-module? Show that M is free if there exists a submodule $N \subseteq M$ such that both N and M/N are free.

Let M and M' be R-modules, and $N \subseteq M$, $N' \subseteq M'$ submodules. Suppose that $N \cong N'$ and $M/N \cong M'/N'$. Determine (by proof or counterexample) which of the following statements holds:

(1) If N is free then $M \cong M'$.

(2) If M/N is free then $M \cong M'$.

12G Geometry

Let $\gamma\colon [a,b] \to S$ be a curve on a smoothly embedded surface $S \subset \mathbf{R}^3$. Define the energy of γ. Show that if γ is a stationary point for the energy for proper variations of γ, then γ satisfies the geodesic equations

$$\frac{d}{dt}(E\dot{\gamma}_1 + F\dot{\gamma}_2) = \frac{1}{2}(E_u\dot{\gamma}_1^2 + 2F_u\dot{\gamma}_1\dot{\gamma}_2 + G_u\dot{\gamma}_2^2)$$

$$\frac{d}{dt}(F\dot{\gamma}_1 + G\dot{\gamma}_2) = \frac{1}{2}(E_v\dot{\gamma}_1^2 + 2F_v\dot{\gamma}_1\dot{\gamma}_2 + G_v\dot{\gamma}_2^2)$$

where $\gamma = (\gamma_1, \gamma_2)$ in terms of a smooth parametrization (u,v) for S, with first fundamental form $E\,du^2 + 2F\,du\,dv + G\,dv^2$.

Now suppose that for every c, d the curves $u = c$, $v = d$ are geodesics.

(i) Show that $(F/\sqrt{G})_v = (\sqrt{G})_u$ and $(F/\sqrt{E})_u = (\sqrt{E})_v$.

(ii) Suppose moreover that the angle between the curves $u = c$, $v = d$ is independent of c and d. Show that $E_v = 0 = G_u$.

13F Analysis II

Explain what it means for two norms on a real vector space to be Lipschitz equivalent. Show that if two norms are Lipschitz equivalent, then one is complete if and only if the other is.

Let $\| - \|$ be an arbitrary norm on the finite-dimensional space $\mathbb{R}^n$, and let $\| - \|_2$ denote the standard (Euclidean) norm. Show that for every $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_2 = 1$, we have

$$\|\mathbf{x}\| \leqslant \|\mathbf{e}_1\| + \|\mathbf{e}_2\| + \cdots + \|\mathbf{e}_n\|$$

where $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ is the standard basis for $\mathbb{R}^n$, and deduce that the function $\| - \|$ is continuous with respect to $\| - \|_2$. Hence show that there exists a constant $m > 0$ such that $\|\mathbf{x}\| \geqslant m$ for all $\mathbf{x}$ with $\|\mathbf{x}\|_2 = 1$, and deduce that $\| - \|$ and $\| - \|_2$ are Lipschitz equivalent.

[*You may assume the Bolzano–Weierstrass Theorem.*]

14F Metric and Topological Spaces

Explain what is meant by a base for a topology. Illustrate your definition by describing bases for the topology induced by a metric on a set, and for the product topology on the cartesian product of two topological spaces.

A topological space $(X, \mathcal{T})$ is said to be *separable* if there is a countable subset $C \subseteq X$ which is dense, i.e. such that $C \cap U \neq \emptyset$ for every nonempty $U \in \mathcal{T}$. Show that a product of two separable spaces is separable. Show also that a metric space is separable if and only if its topology has a countable base, and deduce that every subspace of a separable metric space is separable.

Now let $X = \mathbb{R}$ with the topology $\mathcal{T}$ having as a base the set of all half-open intervals

$$[a, b) = \{x \in \mathbb{R} : a \leqslant x < b\}$$

with $a < b$. Show that X is separable, but that the subspace $Y = \{(x, -x) : x \in \mathbb{R}\}$ of $X \times X$ is not separable.

[*You may assume standard results on countability.*]

15C Complex Methods

Let H be the domain $\mathbb{C} - \{x + iy : x \leq 0, y = 0\}$ (i.e., $\mathbb{C}$ cut along the negative x-axis). Show, by a suitable choice of branch, that the mapping

$$z \mapsto w \;=\; -i \log z$$

maps H onto the strip $S = \{z = x + iy, -\pi < x < \pi\}$.

How would a different choice of branch change the result?

Let G be the domain $\{z \in \mathbb{C} : |z| < 1, |z + i| > \sqrt{2}\}$. Find an analytic transformation that maps G to S, where S is the strip defined above.

16A Methods

Assume $F(x)$ satisfies

$$\int_{-\infty}^{\infty} |F(x)| dx < \infty \,,$$

and that the series

$$g(\tau) = \sum_{n=-\infty}^{\infty} F(2n\pi + \tau)$$

converges uniformly in $[0 \leqslant \tau \leqslant 2\pi]$.

If $\tilde{F}$ is the Fourier transform of F, prove that

$$g(\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \tilde{F}(n) e^{in\tau} \,.$$

[*Hint: prove that* g *is periodic and express its Fourier expansion coefficients in terms of* $\tilde{F}$].

In the case that $F(x) = e^{-|x|}$, evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \,.$$

17C Special Relativity

Write down the formulae for the one-dimensional Lorentz transformation $(x, t) \to (x', t')$ for frames moving with relative velocity v along the x-direction. Derive the relativistic formula for the addition of velocities v and u.

A train, of proper length L, travels past a station at velocity $v > 0$. The origin of the inertial frame S, with coordinates (x, t), in which the train is stationary, is located at the mid-point of the train. The origin of the inertial frame S', with coordinates (x', t'), in which the station is stationary, is located at the mid-point of the platform. Coordinates are chosen such that when the origins coincide then $t = t' = 0$.

Observers A and B, stationary in S, are located, respectively, at the front and rear of the train. Observer C, stationary in S', is located at the origin of S'. At $t' = 0$, C sends two signals, which both travel at speed u, where $v < u \leq c$, one directed towards A and the other towards B, who receive the signals at respective times t_A and t_B. C observes these events to occur, respectively, at times t'_A and t'_B. At $t' = 0$, C also observes that the two ends of the platform coincide with the positions of A and B.

(a) Draw two space-time diagrams, one for S and the other for S', showing the trajectories of the observers and the events that take place.

(b) What is the length of the platform in terms of L? Briefly illustrate your answer by reference to the space-time diagrams.

(c) Calculate the time differences $t_B - t_A$ and $t'_B - t'_A$.

(d) Setting $u = c$, use this example to discuss briefly the fact that two events observed to be simultaneous in one frame need not be observed to be simultaneous in another.

18B Fluid Dynamics

(i) Starting from Euler's equation for an incompressible fluid show that for potential flow with $\mathbf{u} = \boldsymbol{\nabla}\phi$,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + \chi = f(t)\,,$$

where $u = |\mathbf{u}|$, $\chi = p/\rho + V$, the body force per unit mass is $-\boldsymbol{\nabla} V$ and $f(t)$ is an arbitrary function of time.

(ii) Hence show that, for the steady flow of a liquid of density ρ through a pipe of varying cross-section that is subject to a pressure difference $\Delta p = p_1 - p_2$ between its two ends, the mass flow through the pipe per unit time is given by

$$m \equiv \frac{dM}{dt} = S_1 S_2 \sqrt{\frac{2\rho\Delta p}{S_1^2 - S_2^2}}\,,$$

where S_1 and S_2 are the cross-sectional areas of the two ends.

19H Statistics

(i) Consider the linear model

$$Y_i = \alpha + \beta x_i + \varepsilon_i \,,$$

where observations Y_i, $i = 1, \ldots, n$, depend on known explanatory variables x_i, $i = 1, \ldots, n$, and independent $N(0, \sigma^2)$ random variables ε_i, $i = 1, \ldots, n$.

Derive the maximum-likelihood estimators of α, β and σ^2.

Stating clearly any results you require about the distribution of the maximum-likelihood estimators of α, β and σ^2, explain how to construct a test of the hypothesis that $\alpha = 0$ against an unrestricted alternative.

(ii) A simple ballistic theory predicts that the range of a gun fired at angle of elevation θ should be given by the formula

$$Y = \frac{V^2}{g} \sin 2\theta \,,$$

where V is the muzzle velocity, and g is the gravitational acceleration. Shells are fired at 9 different elevations, and the ranges observed are as follows:

θ (degrees)	5	15	25	35	45	55	65	75	85
$\sin 2\theta$	0.1736	0.5	0.7660	0.9397	1	0.9397	0.7660	0.5	0.1736
Y (m)	4322	11898	17485	20664	21296	19491	15572	10027	3458

The model

$$Y_i = \alpha + \beta \sin 2\theta_i + \varepsilon_i \qquad (*)$$

is proposed. Using the theory of part (i) above, find expressions for the maximum-likelihood estimators of α and β.

The t-test of the null hypothesis that $\alpha = 0$ against an unrestricted alternative does not reject the null hypothesis. Would you be willing to accept the model $(*)$? Briefly explain your answer.

[*You may need the following summary statistics of the data. If* $x_i = \sin 2\theta_i$, *then* $\bar{x} \equiv n^{-1} \sum x_i = 0.63986$, $\bar{Y} = 13802$, $S_{xx} \equiv \sum (x_i - \bar{x})^2 = 0.81517$, $S_{xy} = \sum Y_i (x_i - \bar{x}) = 17186$.]

20H Optimization

(i) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$, and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable. Suppose that the problem

$$\max\ f(x) \quad \text{subject to} \ \ g(x) = b$$

is solved by a unique $\bar{x} = \bar{x}(b)$ for each $b \in \mathbb{R}^m$, and that there exists a unique $\lambda(b) \in \mathbb{R}^m$ such that

$$\varphi(b) \ \equiv \ f(\bar{x}(b)) \ = \ \sup_x \{ \ f(x) + \lambda(b)^T (b - g(x)) \ \} \, .$$

Assuming that $\bar{x}$ and λ are continuously differentiable, prove that

$$\frac{\partial \varphi}{\partial b_i}\,(b) \ = \ \lambda_i(b)\,. \qquad (*)$$

(ii) The output of a firm is a function of the capital K deployed, and the amount L of labour employed, given by

$$f(K, L) \ = \ K^\alpha L^\beta \, ,$$

where $\alpha, \beta \in (0, 1)$. The firm's manager has to optimize the output subject to the budget constraint

$$K + wL \ = \ b\,,$$

where $w > 0$ is the wage rate and $b > 0$ is the available budget. By casting the problem in Lagrangian form, find the optimal solution and verify the relation $(*)$.

END OF PAPER