UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 4 June 2008 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

*Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles labelled **A, B, . . . , H** according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Suppose that V and W are finite-dimensional vector spaces over $\mathbb{R}$. What does it mean to say that $\psi : V \to W$ is a linear map? State the rank-nullity formula. Using it, or otherwise, prove that a linear map $\psi : V \to V$ is surjective if, and only if, it is injective.

Suppose that $\psi : V \to V$ is a linear map which has a right inverse, that is to say there is a linear map $\phi : V \to V$ such that $\psi\phi = \mathrm{id}_V$, the identity map. Show that $\phi\psi = \mathrm{id}_V$.

Suppose that A and B are two $n \times n$ matrices over $\mathbb{R}$ such that $AB = I$. Prove that $BA = I$.

2G Groups, Rings and Modules

What does it means to say that a complex number α is algebraic over $\mathbb{Q}$? Define the minimal polynomial of α.

Suppose that α satisfies a nonconstant polynomial $f \in \mathbb{Z}[X]$ which is irreducible over $\mathbb{Z}$. Show that there is an isomorphism $\mathbb{Z}[X]/(f) \cong \mathbb{Z}[\alpha]$.

[*You may assume standard results about unique factorisation, including Gauss's lemma.*]

3F Analysis II

Explain what is meant by the statement that a sequence (f_n) of functions defined on an interval $[a,b]$ converges uniformly to a function f. If (f_n) converges uniformly to f, and each f_n is continuous on $[a,b]$, prove that f is continuous on $[a,b]$.

Now suppose additionally that (x_n) is a sequence of points of $[a,b]$ converging to a limit x. Prove that $f_n(x_n) \to f(x)$.

4F Metric and Topological Spaces

Stating carefully any results on compactness which you use, show that if X is a compact space, Y is a Hausdorff space and $f\colon X \to Y$ is bijective and continuous, then f is a homeomorphism.

Hence or otherwise show that the unit circle $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is homeomorphic to the quotient space $[0,1]/\sim$, where $\sim$ is the equivalence relation defined by

$$x \sim y \Leftrightarrow \text{ either } x = y \text{ or } \{x,y\} = \{0,1\}\ .$$

5D Methods

Describe briefly the method of Lagrange multipliers for finding the stationary values of a function $f(x,y)$ subject to a constraint $g(x,y)=0$.

Use the method to find the largest possible volume of a circular cylinder that has surface area A (including both ends).

6B Electromagnetism

Given the electric potential of a dipole

$$\phi(r,\theta)=\frac{p\cos\theta}{4\pi\epsilon_0 r^2},$$

where p is the magnitude of the dipole moment, calculate the corresponding electric field and show that it can be written as

$$\mathbf{E}(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\left(\mathbf{p}\cdot\hat{\mathbf{e}}_r\right)\hat{\mathbf{e}}_r-\mathbf{p}\right]\ ,$$

where $\hat{\mathbf{e}}_r$ is the unit vector in the radial direction.

7C Special Relativity

A photon of energy E collides with a particle of rest mass m, which is at rest. The final state consists of a photon and a particle of rest mass M, $M>m$. Show that the minimum value of E for which it is possible for this reaction to take place is

$$E_{\min}=\frac{M^2-m^2}{2m}\,c^2\ .$$

8B Fluid Dynamics

(i) Show that for a two-dimensional incompressible flow $(u(x,y),v(x,y),0)$, the vorticity is given by $\boldsymbol{\omega}\equiv\omega_z\hat{\mathbf{e}}_z=(0,0,-\nabla^2\psi)$ where ψ is the stream function.

(ii) Express the z-component of the vorticity equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}=(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$$

in terms of the stream function ψ.

9H Optimization

Goods from three warehouses have to be delivered to five shops, the cost of transporting one unit of good from warehouse i to shop j being c_{ij}, where

$$C = \begin{pmatrix} 2 & 3 & 6 & 6 & 4 \\ 7 & 6 & 1 & 1 & 5 \\ 3 & 6 & 6 & 2 & 1 \end{pmatrix} .$$

The requirements of the five shops are respectively 9, 6, 12, 5 and 10 units of the good, and each warehouse holds a stock of 15 units. Find a minimal-cost allocation of goods from warehouses to shops and its associated cost.

SECTION II

10E Linear Algebra

Define the determinant $\det(A)$ of an $n \times n$ square matrix A over the complex numbers. If A and B are two such matrices, show that $\det(AB) = \det(A)\det(B)$.

Write $p_M(\lambda) = \det(M - \lambda I)$ for the characteristic polynomial of a matrix M. Let A, B, C be $n \times n$ matrices and suppose that C is nonsingular. Show that $p_{BC} = p_{CB}$. Taking $C = A + tI$ for appropriate values of t, or otherwise, deduce that $p_{BA} = p_{AB}$.

Show that if $p_A = p_B$ then $\operatorname{tr}(A) = \operatorname{tr}(B)$. Which of the following statements is true for all $n \times n$ matrices A, B, C? Justify your answers.

(i) $p_{ABC} = p_{ACB}$;

(ii) $p_{ABC} = p_{BCA}$.

11G Groups, Rings and Modules

Let F be a field. Prove that every ideal of the ring $F[X_1, \ldots, X_n]$ is finitely generated.

Consider the set

$$R = \left\{ p(X,Y) = \sum c_{ij} X^i Y^j \in F[X,Y] \;\middle|\; c_{0j} = c_{j0} = 0 \text{ whenever } j > 0 \right\}.$$

Show that R is a subring of $F[X,Y]$ which is not Noetherian.

12G Geometry

Show that the area of a spherical triangle with angles α, β, γ is $\alpha + \beta + \gamma - \pi$. Hence derive the formula for the area of a convex spherical n-gon.

Deduce Euler's formula $F - E + V = 2$ for a decomposition of a sphere into F convex polygons with a total of E edges and V vertices.

A sphere is decomposed into convex polygons, comprising m quadrilaterals, n pentagons and p hexagons, in such a way that at each vertex precisely three edges meet. Show that there are at most 7 possibilities for the pair (m, n), and that at least 3 of these do occur.

13F Analysis II

Let $(u_n(x) : n = 0, 1, 2, \ldots)$ be a sequence of real-valued functions defined on a subset E of $\mathbb{R}$. Suppose that for all n and all $x \in E$ we have $|u_n(x)| \leqslant M_n$, where $\sum_{n=0}^{\infty} M_n$ converges. Prove that $\sum_{n=0}^{\infty} u_n(x)$ converges uniformly on E.

Now let $E = \mathbb{R} \setminus \mathbb{Z}$, and consider the series $\sum_{n=0}^{\infty} u_n(x)$, where $u_0(x) = 1/x^2$ and

$$u_n(x) = 1/(x-n)^2 + 1/(x+n)^2$$

for $n > 0$. Show that the series converges uniformly on $E_R = \{x \in E : |x| < R\}$ for any real number R. Deduce that $f(x) = \sum_{n=0}^{\infty} u_n(x)$ is a continuous function on E. Does the series converge uniformly on E? Justify your answer.

14C Complex Analysis or Complex Methods

Let $f(z) = 1/(e^z - 1)$. Find the first three terms in the Laurent expansion for $f(z)$ valid for $0 < |z| < 2\pi$.

Now let n be a positive integer, and define

$$\begin{aligned} f_1(z) &= \frac{1}{z} + \sum_{r=1}^{n} \frac{2z}{z^2 + 4\pi^2 r^2}\,, \\ f_2(z) &= f(z) - f_1(z)\,. \end{aligned}$$

Show that the singularities of f_2 in $\{z : |z| < 2(n+1)\pi\}$ are all removable. By expanding f_1 as a Laurent series valid for $|z| > 2n\pi$, and f_2 as a Taylor series valid for $|z| < 2(n+1)\pi$, find the coefficients of z^j for $-1 \leq j \leq 1$ in the Laurent series for f valid for $2n\pi < |z| < 2(n+1)\pi$.

By estimating an appropriate integral around the contour $|z| = (2n+1)\pi$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}\,.$$

15D Methods

(a) Legendre's equation may be written in the form

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right)+\lambda y=0.$$

Show that there is a series solution for y of the form

$$y=\sum_{k=0}^{\infty}a_k x^k,$$

where the a_k satisfy the recurrence relation

$$\frac{a_{k+2}}{a_k}=-\frac{(\lambda-k(k+1))}{(k+1)(k+2)}.$$

Hence deduce that there are solutions for $y(x)=P_n(x)$ that are polynomials of degree n, provided that $\lambda=n(n+1)$. Given that a_0 is then chosen so that $P_n(1)=1$, find the explicit form for $P_2(x)$.

(b) Laplace's equation for $\Phi(r,\theta)$ in spherical polar coordinates (r,θ,ϕ) may be written in the axisymmetric case as

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial\Phi}{\partial x}\right)=0,$$

where $x=\cos\theta$.

Write down without proof the general form of the solution obtained by the method of separation of variables. Use it to find the form of Φ exterior to the sphere $r=a$ that satisfies the boundary conditions, $\Phi(a,x)=1+x^2$, and $\lim_{r\to\infty}\ \Phi(r,x)=0$.

16A Quantum Mechanics

Give the physical interpretation of the expression

$$\langle A \rangle_\psi = \int \psi(x)^* \hat{A} \psi(x) dx$$

for an observable A, where $\hat{A}$ is a Hermitian operator and ψ is normalised. By considering the norm of the state $(A + i\lambda B)\psi$ for two observables A and B, and real values of λ, show that

$$\langle A^2 \rangle_\psi \langle B^2 \rangle_\psi \geqslant \frac{1}{4} |\langle [A, B] \rangle_\psi|^2 \,.$$

Deduce the uncertainty relation

$$\Delta A \Delta B \geqslant \frac{1}{2} |\langle [A, B] \rangle_\psi| \,,$$

where ΔA is the uncertainty of A.

A particle of mass m moves in one dimension under the influence of potential $\frac{1}{2} m\omega^2 x^2$. By considering the commutator $[x, p]$, show that the expectation value of the Hamiltonian satisfies

$$\langle H \rangle_\psi \geqslant \frac{1}{2} \hbar\omega \,.$$

17B Electromagnetism

Two perfectly conducting rails are placed on the xy-plane, one coincident with the x-axis, starting at $(0,0)$, the other parallel to the first rail a distance ℓ apart, starting at $(0,\ell)$. A resistor R is connected across the rails between $(0,0)$ and $(0,\ell)$, and a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_z$ is the unit vector along the z-axis and $B > 0$, fills the entire region of space. A metal bar of negligible resistance and mass m slides without friction on the two rails, lying perpendicular to both of them in such a way that it closes the circuit formed by the rails and the resistor. The bar moves with speed v to the right such that the area of the loop becomes larger with time.

(i) Calculate the current in the resistor and indicate its direction of flow in a diagram of the system.

(ii) Show that the magnetic force on the bar is

$$\mathbf{F} = -\frac{B^2\ell^2 v}{R}\hat{\mathbf{e}}_x \ .$$

(iii) Assume that the bar starts moving with initial speed v_0 at time $t = 0$, and is then left to slide freely. Using your result from part (ii) and Newton's laws show that its velocity at the time t is

$$v(t) = v_0 e^{-(B^2\ell^2/mR)t}.$$

(iv) By calculating the total energy delivered to the resistor, verify that energy is conserved.

18D Numerical Analysis

(a) A Householder transformation (reflection) is given by

$$H = I - \frac{2uu^T}{\|u\|^2},$$

where $H \in \mathbb{R}^{m \times m}, u \in \mathbb{R}^m$, and I is the $m \times m$ unit matrix and u is a non-zero vector which has norm $\|u\| = (\sum_{i=1}^m u_i^2)^{1/2}$. Show that H is orthogonal.

(b) Suppose that $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ with $n < m$. Show that if x minimises $\|Ax - b\|^2$ then it also minimises $\|QAx - Qb\|^2$, where Q is an arbitrary $m \times m$ orthogonal matrix.

(c) Using Householder reflection, find the x that minimises $\|Ax - b\|^2$ when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 0 & 2 \\ 0 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}.$$

19H Statistics

Suppose that the joint distribution of random variables X, Y taking values in $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ is given by the joint probability generating function

$$\varphi(s,t) \equiv E\left[s^X t^Y\right] = \frac{1-\alpha-\beta}{1-\alpha s-\beta t},$$

where the unknown parameters α and β are positive, and satisfy the inequality $\alpha + \beta < 1$. Find $E(X)$. Prove that the probability mass function of (X, Y) is

$$f(x, y \,|\, \alpha, \beta) = (1-\alpha-\beta)\binom{x+y}{x}\alpha^x\beta^y \qquad (x, y \in \mathbb{Z}^+),$$

and prove that the maximum-likelihood estimators of α and β based on a sample of size n drawn from the distribution are

$$\hat{\alpha} = \frac{\overline{X}}{1+\overline{X}+\overline{Y}}, \qquad \hat{\beta} = \frac{\overline{Y}}{1+\overline{X}+\overline{Y}},$$

where $\overline{X}$ (respectively, $\overline{Y}$) is the sample mean of $X_1, \dots, X_n$ (respectively, $Y_1, \dots, Y_n$).

By considering $\hat{\alpha} + \hat{\beta}$ or otherwise, prove that the maximum-likelihood estimator is biased. Stating clearly any results to which you appeal, prove that as $n \to \infty$, $\hat{\alpha} \to \alpha$, making clear the sense in which this convergence happens.

20H Markov Chains

A Markov chain with state–space $I = \mathbb{Z}^+$ has non-zero transition probabilities $p_{00} = q_0$ and

$$p_{i,i+1} = p_i \,, \quad p_{i+1,i} = q_{i+1} \qquad (i \in I)\,.$$

Prove that this chain is recurrent if and only if

$$\sum_{n \geqslant 1} \prod_{r=1}^{n} \frac{q_r}{p_r} = \infty\,.$$

Prove that this chain is positive-recurrent if and only if

$$\sum_{n \geqslant 1} \prod_{r=1}^{n} \frac{p_{r-1}}{q_r} < \infty\,.$$

END OF PAPER