UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday 3 June 2008 9.00 to 12.00

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B,*** *. . . ,* ***H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Let A be an $n \times n$ matrix over $\mathbb{C}$. What does it mean to say that λ is an eigenvalue of A? Show that A has at least one eigenvalue. For each of the following statements, provide a proof or a counterexample as appropriate.

(i) If A is Hermitian, all eigenvalues of A are real.

(ii) If all eigenvalues of A are real, A is Hermitian.

(iii) If all entries of A are real and positive, all eigenvalues of A have positive real part.

(iv) If A and B have the same trace and determinant then they have the same eigenvalues.

2G Geometry

Show that any element of $SO(3,\mathbb{R})$ is a rotation, and that it can be written as the product of two reflections.

3C Complex Analysis or Complex Methods

Given that $f(z)$ is an analytic function, show that the mapping $w = f(z)$

(a) preserves angles between smooth curves intersecting at z if $f'(z) \neq 0$;

(b) has Jacobian given by $|f'(z)|^2$.

4C Special Relativity

In an inertial frame S a photon of energy E is observed to travel at an angle θ relative to the x-axis. The inertial frame S' moves relative to S at velocity v in the x-direction and the x'-axis of S' is taken parallel to the x-axis of S. Observed in S', the photon has energy E' and travels at an angle θ' relative to the x'-axis. Show that

$$E' = \frac{E(1-\beta\cos\theta)}{\sqrt{1-\beta^2}}, \qquad \cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta},$$

where $\beta = v/c$.

5B Fluid Dynamics

Verify that the two-dimensional flow given in Cartesian coordinates by

$$\mathbf{u} = (\mathrm{e}^y \sinh x, -\mathrm{e}^y \cosh x)$$

satisfies $\nabla \cdot \mathbf{u} = 0$. Find the stream function $\psi(x, y)$. Sketch the streamlines.

6D Numerical Analysis

Show that if $A = LDL^T$, where $L \in \mathbb{R}^{m \times m}$ is a lower triangular matrix with all elements on the main diagonal being unity and $D \in \mathbb{R}^{m \times m}$ is a diagonal matrix with positive elements, then A is positive definite. Find L and the corresponding D when

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

7H Statistics

A Bayesian statistician observes a random sample $X_1, \ldots, X_n$ drawn from a $N(\mu, \tau^{-1})$ distribution. He has a prior density for the unknown parameters μ, τ of the form

$$\pi_0(\mu, \tau) \propto \tau^{\alpha_0 - 1} \exp\left(-\tfrac{1}{2} K_0 \tau (\mu - \mu_0)^2 - \beta_0 \tau\right) \sqrt{\tau},$$

where α_0, β_0, μ_0 and K_0 are constants which he chooses. Show that after observing $X_1, \ldots, X_n$ his posterior density $\pi_n(\mu, \tau)$ is again of the form

$$\pi_n(\mu, \tau) \propto \tau^{\alpha_n - 1} \exp\left(-\tfrac{1}{2} K_n \tau (\mu - \mu_n)^2 - \beta_n \tau\right) \sqrt{\tau},$$

where you should find explicitly the form of α_n, β_n, μ_n and K_n.

8H Optimization

State the Lagrangian Sufficiency Theorem for the maximization over x of $f(x)$ subject to the constraint $g(x) = b$.

For each $p > 0$, solve

$$\max \sum_{i=1}^{d} x_i^p \quad \text{subject to} \quad \sum_{i=1}^{d} x_i = 1, \quad x_i \geqslant 0.$$

SECTION II

9E Linear Algebra

Let A be an $m \times n$ matrix of real numbers. Define the row rank and column rank of A and show that they are equal.

Show that if a matrix A' is obtained from A by elementary row and column operations then $\operatorname{rank}(A') = \operatorname{rank}(A)$.

Let P, Q and R be $n \times n$ matrices. Show that the $2n \times 2n$ matrices $\begin{pmatrix} PQ & 0 \\ Q & QR \end{pmatrix}$ and $\begin{pmatrix} 0 & PQR \\ Q & 0 \end{pmatrix}$ have the same rank.

Hence, or otherwise, prove that

$$\operatorname{rank}(PQ) + \operatorname{rank}(QR) \leqslant \operatorname{rank}(Q) + \operatorname{rank}(PQR).$$

10G Groups, Rings and Modules

(i) Show that A_4 is not simple.

(ii) Show that the group $\operatorname{Rot}(D)$ of rotational symmetries of a regular dodecahedron is a simple group of order 60.

(iii) Show that $\operatorname{Rot}(D)$ is isomorphic to A_5.

11F Analysis II

State and prove the Contraction Mapping Theorem.

Let (X, d) be a nonempty complete metric space and $f\colon X \to X$ a mapping such that, for some $k > 0$, the kth iterate f^k of f (that is, f composed with itself k times) is a contraction mapping. Show that f has a unique fixed point.

Now let X be the space of all continuous real-valued functions on $[0, 1]$, equipped with the uniform norm $\|h\|_\infty = \sup\{|h(t)| : t \in [0, 1]\}$, and let $\phi\colon \mathbb{R} \times [0, 1] \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition

$$|\phi(x, t) - \phi(y, t)| \leqslant M|x - y|$$

for all $t \in [0, 1]$ and all $x, y \in \mathbb{R}$, where M is a constant. Let $F\colon X \to X$ be defined by

$$F(h)(t) = g(t) + \int_0^t \phi(h(s), s)\ ds\ ,$$

where g is a fixed continuous function on $[0, 1]$. Show by induction on n that

$$|F^n(h)(t) - F^n(k)(t)| \leqslant \frac{M^n t^n}{n!}\|h - k\|_\infty$$

for all $h, k \in X$ and all $t \in [0, 1]$. Deduce that the integral equation

$$f(t) = g(t) + \int_0^t \phi(f(s), s)\ ds$$

has a unique continuous solution f on $[0, 1]$.

12F Metric and Topological Spaces

Write down the definition of a topology on a set X.

For each of the following families $\mathcal{T}$ of subsets of $\mathbb{Z}$, determine whether $\mathcal{T}$ is a topology on $\mathbb{Z}$. In the cases where the answer is 'yes', determine also whether $(\mathbb{Z}, \mathcal{T})$ is a Hausdorff space and whether it is compact.

(*a*) $\mathcal{T} = \{U \subseteq \mathbb{Z} :$ either U is finite or $0 \in U\}$.

(*b*) $\mathcal{T} = \{U \subseteq \mathbb{Z} :$ either $\mathbb{Z} \setminus U$ is finite or $0 \notin U\}$.

(*c*) $\mathcal{T} = \{U \subseteq \mathbb{Z} :$ there exists $k > 0$ such that, for all n, $n \in U \Leftrightarrow n + k \in U\}$.

(*d*) $\mathcal{T} = \{U \subseteq \mathbb{Z} :$ for all $n \in U$, there exists $k > 0$ such that $\{n + km : m \in \mathbb{Z}\} \subseteq U\}$.

13C Complex Analysis or Complex Methods

By a suitable choice of contour show the following:

(a)

$$\int_0^\infty \frac{x^{1/n}}{1+x^2}\,dx \;=\; \frac{\pi}{2\cos(\pi/2n)}\,,$$

where $n > 1$,

(b)

$$\int_0^\infty \frac{x^{1/2}\log x}{1+x^2}\,dx \;=\; \frac{\pi^2}{2\sqrt{2}}\,.$$

14D Methods

Write down the Euler–Lagrange equation for the variational problem for $y(x)$ that extremizes the integral I defined as

$$I = \int_{x_1}^{x_2} f(x,y,y')dx,$$

with boundary conditions $y(x_1) = y_1, y(x_2) = y_2$, where y_1 and y_2 are positive constants such that $y_2 > y_1$, with $x_2 > x_1$. Find a first integral of the equation when f is independent of y, i.e. $f = f(x,y')$.

A light ray moves in the (x,y) plane from (x_1,y_1) to (x_2,y_2) with speed $c(x)$ taking a time T. Show that the equation of the path that makes T an extremum satisfies

$$\frac{dy}{dx} = \frac{c(x)}{\sqrt{k^2 - c^2(x)}},$$

where k is a constant and write down an integral relating k, x_1, x_2, y_1 and y_2.

When $c(x) = ax$ where a is a constant and $k = ax_2$, show that the path is given by

$$(y_2 - y)^2 = x_2^2 - x^2.$$

15A Quantum Mechanics

The radial wavefunction $g(r)$ for the hydrogen atom satisfies the equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dg(r)}{dr}\right)-\frac{e^2 g(r)}{4\pi\epsilon_0 r}+\hbar^2\frac{\ell(\ell+1)}{2mr^2}g(r)=Eg(r)\,. \qquad (*)$$

With reference to the general form for the time-independent Schrödinger equation, explain the origin of each term. What are the allowed values of ℓ?

The lowest-energy bound-state solution of $(*)$, for given ℓ, has the form $r^\alpha e^{-\beta r}$. Find α and β and the corresponding energy E in terms of ℓ.

A hydrogen atom makes a transition between two such states corresponding to $\ell+1$ and ℓ. What is the frequency of the emitted photon?

16B Electromagnetism

Suppose that the current density $\mathbf{J}(\mathbf{r})$ is constant in time but the charge density $\rho(\mathbf{r}, t)$ is not.

(i) Show that ρ is a linear function of time:

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) + \dot{\rho}(\mathbf{r}, 0)t,$$

where $\dot{\rho}(\mathbf{r}, 0)$ is the time derivative of ρ at time $t = 0$.

(ii) The magnetic induction due to a current density $\mathbf{J}(\mathbf{r})$ can be written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' \ .$$

Show that this can also be written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'. \qquad (1)$$

(iii) Assuming that $\mathbf{J}$ vanishes at infinity, show that the curl of the magnetic field in (1) can be written as

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) + \frac{\mu_0}{4\pi} \nabla \int \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \ . \qquad (2)$$

[*You may find useful the identities* $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ *and also* $\nabla^2 \left(1/|\mathbf{r} - \mathbf{r}'|\right) = -4\pi\delta(\mathbf{r} - \mathbf{r}')$.]

(iv) Show that the second term on the right hand side of (2) can be expressed in terms of the time derivative of the electric field in such a way that $\mathbf{B}$ itself obeys Ampère's law with Maxwell's displacement current term, i.e. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E} / \partial t$.

17B Fluid Dynamics

Two incompressible fluids flow in infinite horizontal streams, the plane of contact being $z = 0$, with z positive upwards. The flow is given by

$$\mathbf{U}(\mathbf{r}) = \begin{cases} U_2 \hat{\mathbf{e}}_x, & z > 0; \\ U_1 \hat{\mathbf{e}}_x, & z < 0, \end{cases}$$

where $\hat{\mathbf{e}}_x$ is the unit vector in the positive x direction. The upper fluid has density ρ_2 and pressure $p_0 - g\rho_2 z$, the lower has density ρ_1 and pressure $p_0 - g\rho_1 z$, where p_0 is a constant and g is the acceleration due to gravity.

(i) Consider a perturbation to the flat surface $z = 0$ of the form

$$z \equiv \zeta(x, y, t) = \zeta_0 e^{i(kx+\ell y)+st} \ .$$

State the kinematic boundary conditions on the velocity potentials ϕ_i that hold on the interface in the two domains, and show by linearising in ζ that they reduce to

$$\frac{\partial \phi_i}{\partial z} = \frac{\partial \zeta}{\partial t} + U_i \frac{\partial \zeta}{\partial x} \quad (z = 0, \ i = 1, 2) \, .$$

(ii) State the dynamic boundary condition on the perturbed interface, and show by linearising in ζ that it reduces to

$$\rho_1 \left(U_1 \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial t} + g\zeta \right) = \rho_2 \left(U_2 \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial t} + g\zeta \right) \quad (z = 0) \, .$$

(iii) Use the velocity potentials

$$\phi_1 = U_1 x + A_1 e^{qz} e^{i(kx+\ell y)+st} \ , \quad \phi_2 = U_2 x + A_2 e^{-qz} e^{i(kx+\ell y)+st},$$

where $q = \sqrt{k^2 + \ell^2}$,and the conditions in (i) and (ii) to perform a stability analysis. Show that the relation between s, k and ℓ is

$$s = -ik \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{k^2 \rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} - \frac{qg(\rho_1 - \rho_2)}{\rho_1 + \rho_2} \right]^{1/2} \, .$$

Find the criterion for instability.

18H Statistics

Suppose that $X_1, \ldots, X_n$ is a sample of size n with common $N(\mu_X, 1)$ distribution, and $Y_1, \ldots, Y_n$ is an independent sample of size n from a $N(\mu_Y, 1)$ distribution.

(i) Find (with careful justification) the form of the size-α likelihood–ratio test of the null hypothesis $H_0 : \mu_Y = 0$ against alternative $H_1 : (\mu_X, \mu_Y)$ unrestricted.

(ii) Find the form of the size-α likelihood–ratio test of the hypothesis

$$H_0 : \mu_X \geqslant A, \mu_Y = 0\,,$$

against $H_1 : (\mu_X, \mu_Y)$ unrestricted, where A is a given constant.

Compare the critical regions you obtain in (i) and (ii) and comment briefly.

19H Markov Chains

The village green is ringed by a fence with N fenceposts, labelled $0, 1, \ldots, N-1$. The village idiot is given a pot of paint and a brush, and started at post 0 with instructions to paint all the posts. He paints post 0, and then chooses one of the two nearest neighbours, 1 or $N-1$, with equal probability, moving to the chosen post and painting it. After painting a post, he chooses with equal probability one of the two nearest neighbours, moves there and paints it (regardless of whether it is already painted). Find the distribution of the last post unpainted.

END OF PAPER