UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 3 June 2008 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***C*** *and* ***E*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups

Define the *signature* $\epsilon(\sigma)$ of a permutation $\sigma \in S_n$, and show that the map $\epsilon : S_n \to \{-1, 1\}$ is a homomorphism.

Define the *alternating group* A_n, and prove that it is a subgroup of S_n. Is A_n a normal subgroup of S_n? Justify your answer.

2E Groups

What is the *orthogonal group* $O(n)$? What is the *special orthogonal group* $SO(n)$?

Show that every element of the special orthogonal group $SO(3)$ has an eigenvector with eigenvalue 1. Is this also true for every element of the orthogonal group $O(3)$? Justify your answer.

3C Vector Calculus

A curve is given in terms of a parameter t by

$$\mathbf{x}(t) = (t - \tfrac{1}{3}t^3\,,\, t^2\,,\, t + \tfrac{1}{3}t^3)\,.$$

(i) Find the arc length of the curve between the points with $t = 0$ and $t = 1$.

(ii) Find the unit tangent vector at the point with parameter t, and show that the principal normal is orthogonal to the z direction at each point on the curve.

4C Vector Calculus

What does it mean to say that T_{ij} transforms as a second rank tensor?

If T_{ij} transforms as a second rank tensor, show that $\dfrac{\partial T_{ij}}{\partial x_j}$ transforms as a vector.

SECTION II

5E Groups

For a normal subgroup H of a group G, explain carefully how to make the set of (left) cosets of H into a group.

For a subgroup H of a group G, show that the following are equivalent:

(i) H is a normal subgroup of G;

(ii) there exist a group K and a homomorphism $\theta : G \to K$ such that H is the kernel of θ.

Let G be a finite group that has a proper subgroup H of index n (in other words, $|H| = |G|/n$). Show that if $|G| > n!$ then G cannot be simple. [Hint: Let G act on the set of left cosets of H by left multiplication.]

6E Groups

Prove that two elements of S_n are conjugate if and only if they have the same cycle type.

Describe (without proof) a necessary and sufficient condition for a permutation $\sigma \in A_n$ to have the same conjugacy class in A_n as it has in S_n.

For which $\sigma \in S_n$ is σ conjugate (in S_n) to σ^2 ?

For every $\sigma \in A_5$, show that σ is conjugate to σ^{-1} (in A_5). Exhibit a positive integer n and a $\sigma \in A_n$ such that σ is not conjugate to σ^{-1} (in A_n).

7E Groups

Show that every Möbius map may be expressed as a composition of maps of the form $z \mapsto z + a$, $z \mapsto \lambda z$ and $z \mapsto 1/z$ (where a and λ are complex numbers).

Which of the following statements are true and which are false? Justify your answers.

(i) Every Möbius map that fixes ∞ may be expressed as a composition of maps of the form $z \mapsto z + a$ and $z \mapsto \lambda z$ (where a and λ are complex numbers).

(ii) Every Möbius map that fixes 0 may be expressed as a composition of maps of the form $z \mapsto \lambda z$ and $z \mapsto 1/z$ (where λ is a complex number).

(iii) Every Möbius map may be expressed as a composition of maps of the form $z \mapsto z + a$ and $z \mapsto 1/z$ (where a is a complex number).

8E Groups

State and prove the orbit–stabilizer theorem. Deduce that if x is an element of a finite group G then the order of x divides the order of G.

Prove Cauchy's theorem, that if p is a prime dividing the order of a finite group G then G contains an element of order p.

For which positive integers n does there exist a group of order n in which every element (apart from the identity) has order 2?

Give an example of an infinite group in which every element (apart from the identity) has order 2.

9C Vector Calculus

Let $\mathbf{F} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})$, where $\mathbf{x}$ is the position vector and $\boldsymbol{\omega}$ is a uniform vector field.

(i) Use the divergence theorem to evaluate the surface integral $\int_S \mathbf{F} \cdot d\mathbf{S}$, where S is the closed surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

(ii) Show that $\boldsymbol{\nabla} \times \mathbf{F} = 0$. Show further that the scalar field ϕ given by

$$\phi = \tfrac{1}{2}\,(\boldsymbol{\omega} \cdot \mathbf{x})^2 - \tfrac{1}{2}\,(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{x} \cdot \mathbf{x})$$

satisfies $\mathbf{F} = \boldsymbol{\nabla}\phi$. Describe geometrically the surfaces of constant ϕ.

10C Vector Calculus

Find the effect of a rotation by $\pi/2$ about the z-axis on the tensor

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}.$$

Hence show that the most general isotropic tensor of rank 2 is $\lambda\delta_{ij}$, where λ is an arbitrary scalar.

Prove that there is no non-zero isotropic vector, and write down without proof the most general isotropic tensor of rank 3.

Deduce that if T_{ijkl} is an isotropic tensor then the following results hold, for some scalars μ and ν:

(i) $\epsilon_{ijk}\, T_{ijkl} \;\; = 0\,;$

(ii) $\delta_{ij}\, T_{ijkl} \;\; = \mu\, \delta_{kl}\,;$

(iii) $\epsilon_{ijm}\, T_{ijkl} \; = \nu\, \epsilon_{klm}\,.$

Verify these three results in the case $T_{ijkl} \; = \; \alpha\, \delta_{ij}\, \delta_{kl} + \beta\, \delta_{ik}\, \delta_{jl} + \gamma\, \delta_{il}\, \delta_{jk}$, expressing μ and ν in terms of α, β and γ .

11C Vector Calculus

Let V be a volume in $\mathbb{R}^3$ bounded by a closed surface S .

(a) Let f and g be twice differentiable scalar fields such that $f = 1$ on S and $\nabla^2 g = 0$ in V . Show that

$$\int_V \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g \, dV \; = \; 0\,.$$

(b) Let V be the sphere $|\mathbf{x}| \leqslant a$. Evaluate the integral

$$\int_V \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, dV$$

in the cases where u and v are given in spherical polar coordinates by:

(i) $u \; = \; r\,, \qquad v \; = \; r\cos\theta\,;$

(ii) $u \; = \; r/a\,, \quad v \; = \; r^2 \cos^2\theta\,;$

(iii) $u \; = \; r/a\,, \quad v \; = \; 1/r\,.$

Comment on your results in the light of part (a).

12C Vector Calculus

Let A be the closed planar region given by

$$y \leqslant x \leqslant 2y, \quad \frac{1}{y} \leqslant x \leqslant \frac{2}{y}\,.$$

(i) Evaluate by means of a suitable change of variables the integral

$$\int_A \frac{x}{y}\,dx\,dy\,.$$

(ii) Let C be the boundary of A. Evaluate the line integral

$$\oint_C \frac{x^2}{2y}dy - dx$$

by integrating along each section of the boundary.

(iii) Comment on your results.

END OF PAPER