UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 30 May 2008 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Differential Equations

Let a be a positive constant. Find the solution to the differential equation

$$\frac{d^4y}{dx^4} - a^4 y = \mathrm{e}^{-ax}$$

that satisfies $y(0) = 1$ and $y \to 0$ as $x \to \infty$.

2A Differential Equations

Find the fixed points of the difference equation

$$u_{n+1} = \lambda u_n (1 - u_n^2) .$$

Show that a stable fixed point exists when $-1 < \lambda < 1$ and also when $1 < \lambda < 2$.

3F Probability

There are n socks in a drawer, three of which are red and the rest black. John chooses his socks by selecting two at random from the drawer and puts them on. He is three times more likely to wear socks of different colours than to wear matching red socks. Find n.

For this value of n, what is the probability that John wears matching black socks?

4F Probability

A standard six-sided die is thrown. Calculate the mean and variance of the number shown.

The die is thrown n times. By using Chebyshev's inequality, find an n such that

$$\mathbb{P}\left(\left| \frac{Y_n}{n} - 3.5 \right| > 1.5 \right) \leqslant 0.1$$

where Y_n is the total of the numbers shown over the n throws.

SECTION II

5A Differential Equations

Two cups of hot tea at temperatures $T_1(t)$ and $T_2(t)$ cool in a room at ambient constant temperature T_∞. Initially $T_1(0) = T_2(0) = T_0 > T_\infty$.

Cup 1 has cool milk added instantaneously at $t = 1$; in contrast, cup 2 has cool milk added at a constant rate for $1 \leqslant t \leqslant 2$. Briefly explain the use of the differential equations

$$\frac{dT_1}{dt} = -a\,(T_1 - T_\infty) - \delta\,(t-1)\,,$$

$$\frac{dT_2}{dt} = -a\,(T_2 - T_\infty) - H(t-1) + H(t-2)\,,$$

where $\delta(t)$ and $H(t)$ are the Dirac delta and Heaviside functions respectively, and a is a positive constant.

(i) Show that for $0 \leqslant t < 1$

$$T_1(t) = T_2(t) = T_\infty + (\,T_0 - T_\infty)\,\mathrm{e}^{-at}.$$

(ii) Determine the jump (discontinuity) condition for T_1 at $t = 1$ and hence find $T_1(t)$ for $t > 1$.

(iii) Using continuity of $T_2(t)$ at $t = 1$ show that for $1 < t < 2$

$$T_2(t) \;=\; T_\infty - \frac{1}{a} + \mathrm{e}^{-at}\left(T_0 - T_\infty + \frac{1}{a}\,\mathrm{e}^{a}\right).$$

(iv) Compute $T_2(t)$ for $t > 2$ and show that for $t > 2$

$$T_1(t) - T_2(t) \;=\; \left(\frac{1}{a}\,\mathrm{e}^{a} - 1 - \frac{1}{a}\right)\mathrm{e}^{\,(1-t)a}.$$

(v) Find the time t^*, after $t = 1$, at which $T_1 = T_2$.

6A Differential Equations

The linear second-order differential equation

$$\frac{d^2y}{dx^2} + p(x)\,\frac{dy}{dx} + q(x)\,y \;=\; 0$$

has linearly independent solutions $y_1(x)$ and $y_2(x)$. Define the *Wronskian* W of $y_1(x)$ and $y_2(x)$.

Suppose that $y_1(x)$ is known. Use the Wronskian to write down a first-order differential equation for $y_2(x)$. Hence express $y_2(x)$ in terms of $y_1(x)$ and W.

Show further that W satisfies the differential equation

$$\frac{dW}{dx} + p(x)\,W \;=\; 0\,.$$

Verify that $y_1(x) = x^2 - 2x + 1$ is a solution of

$$(x-1)^2\,\frac{d^2y}{dx^2} + (x-1)\,\frac{dy}{dx} - 4y \;=\; 0\,. \qquad (*)$$

Compute the Wronskian and hence determine a second, linearly independent, solution of $(*)$.

7A Differential Equations

Find the first three non-zero terms in series solutions $y_1(x)$ and $y_2(x)$ for the differential equation

$$x\,\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4\,x^3 y \;=\; 0\,, \qquad (*)$$

that satisfy the boundary conditions

$$\begin{aligned} y_1(0) \;&=\; a\,, \quad & y_1''(0) \;&=\; 0\,, \\ y_2(0) \;&=\; 0\,, \quad & y_2''(0) \;&=\; b\,, \end{aligned}$$

where a and b are constants.

Determine the value of α such that the change of variable $u = x^\alpha$ transforms $(*)$ into a differential equation with constant coefficients. Hence find the general solution of $(*)$.

8A Differential Equations

Consider the function

$$f(x,y) \;=\; x^2 + y^2 - \frac{1}{2}\,x^4 - b\,x^2y^2 - \frac{1}{2}\,y^4\,,$$

where b is a positive constant.

Find the critical points of $f(x,y)$, assuming $b \neq 1$. Determine the type of each critical point and sketch contours of constant $f(x,y)$ in the two cases (i) $b < 1$ and (ii) $b > 1$.

For $b = 1$ describe the subset of the (x,y) plane on which $f(x,y)$ attains its maximum value.

9F Probability

A population evolves in generations. Let Z_n be the number of members in the nth generation, with $Z_0 = 1$. Each member of the nth generation gives birth to a family, possibly empty, of members of the $(n+1)$th generation; the size of this family is a random variable and we assume that the family sizes of all individuals form a collection of independent identically distributed random variables each with generating function G.

Let G_n be the generating function of Z_n. State and prove a formula for G_n in terms of G. Determine the mean of Z_n in terms of the mean of Z_1.

Suppose that Z_1 has a Poisson distribution with mean λ. Find an expression for x_{n+1} in terms of x_n, where $x_n = \mathbb{P}\{Z_n = 0\}$ is the probability that the population becomes extinct by the nth generation.

10F Probability

A and B play a series of games. The games are independent, and each is won by A with probability p and by B with probability $1-p$. The players stop when the number of wins by one player is three greater than the number of wins by the other player. The player with the greater number of wins is then declared overall winner.

(i) Find the probability that exactly 5 games are played.

(ii) Find the probability that A is the overall winner.

11F Probability

Let X and Y have the bivariate normal density function

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(x^2-2\rho xy+y^2\right)\right\}, \qquad x,y \in \mathbb{R},$$

for fixed $\rho \in (-1,1)$. Let $Z = (Y-\rho X)/\sqrt{1-\rho^2}$. Show that X and Z are independent $N(0,1)$ variables. Hence, or otherwise, determine

$$\mathbb{P}(X > 0, Y > 0).$$

12F Probability

The discrete random variable Y has distribution given by

$$\mathbb{P}(Y = k) = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots,$$

where $p \in (0,1)$. Determine the mean and variance of Y.

A fair die is rolled until all 6 scores have occurred. Find the mean and standard deviation of the number of rolls required.

[Hint: $\displaystyle\sum_{i=1}^{6}\left(\frac{6}{i}\right)^2 = 53.7$]

END OF PAPER