UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 29 May 2008 9.00 to 12.00

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Vectors and Matrices

State de Moivre's Theorem. By evaluating

$$\sum_{r=1}^{n} e^{ir\theta} ,$$

or otherwise, show that

$$\sum_{r=1}^{n} \cos(r\theta) = \frac{\cos(n\theta) - \cos((n+1)\theta)}{2(1-\cos\theta)} - \frac{1}{2} .$$

Hence show that

$$\sum_{r=1}^{n} \cos\left(\frac{2p\pi r}{n+1}\right) = -1 ,$$

where p is an integer in the range $1 \leqslant p \leqslant n$.

2A Vectors and Matrices

Let U be an $n \times n$ unitary matrix ($U^\dagger U = UU^\dagger = I$). Suppose that A and B are $n \times n$ Hermitian matrices such that $U = A + iB$.

Show that

(i) A and B commute,

(ii) $A^2 + B^2 = I$.

Find A and B in terms of U and $U^\dagger$, and hence show that A and B are uniquely determined for a given U.

3F Analysis I

State the ratio test for the convergence of a series.

Find all real numbers x such that the series

$$\sum_{n=1}^{\infty} \frac{x^n - 1}{n}$$

converges.

4E Analysis I

Let $f:\ [0,1] \to \mathbb{R}$ be Riemann integrable, and for $0 \leqslant x \leqslant 1$ set $F(x) = \int_0^x f(t)\,dt$.

Assuming that f is continuous, prove that for every $0 < x < 1$ the function F is differentiable at x, with $F'(x) = f(x)$.

If we do not assume that f is continuous, must it still be true that F is differentiable at every $0 < x < 1$? Justify your answer.

SECTION II

5B Vectors and Matrices

(a) Use suffix notation to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}\ .$$

Hence, or otherwise, expand

(i) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$,

(ii) $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

(b) Write down the equation of the line that passes through the point $\mathbf{a}$ and is parallel to the unit vector $\hat{\mathbf{t}}$.

The lines L_1 and L_2 in three dimensions pass through $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively and are parallel to the unit vectors $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$ respectively. Show that a necessary condition for L_1 and L_2 to intersect is

$$(\mathbf{a}_1 - \mathbf{a}_2) \cdot \left(\hat{\mathbf{t}}_1 \times \hat{\mathbf{t}}_2\right) = 0\ .$$

Why is this condition not sufficient?

In the case in which L_1 and L_2 are non-parallel and non-intersecting, find an expression for the shortest distance between them.

6A Vectors and Matrices

A real 3×3 matrix A with elements A_{ij} is said to be *upper triangular* if $A_{ij} = 0$ whenever $i > j$. Prove that if A and B are upper triangular 3×3 real matrices then so is the matrix product AB.

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} .$$

Show that $A^3 + A^2 - A = I$. Write A^{-1} as a linear combination of A^2, A and I and hence compute A^{-1} explicitly.

For all integers n (including negative integers), prove that there exist coefficients α_n, β_n and γ_n such that

$$A^n = \alpha_n A^2 + \beta_n A + \gamma_n I .$$

For all integers n (including negative integers), show that

$$(A^n)_{11} = 1 , \qquad (A^n)_{22} = (-1)^n , \qquad \text{and} \quad (A^n)_{23} = n(-1)^{n-1}.$$

Hence derive a set of 3 simultaneous equations for $\{\alpha_n, \beta_n, \gamma_n\}$ and find their solution.

7C Vectors and Matrices

Prove that any n orthonormal vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Let A be a real symmetric $n \times n$ matrix with n orthonormal eigenvectors $\mathbf{e}_i$ and corresponding eigenvalues λ_i. Obtain coefficients a_i such that

$$\mathbf{x} = \sum_i a_i \, \mathbf{e}_i$$

is a solution to the equation

$$A\mathbf{x} - \mu\mathbf{x} = \mathbf{f},$$

where $\mathbf{f}$ is a given vector and μ is a given scalar that is not an eigenvalue of A.

How would your answer differ if $\mu = \lambda_1$?

Find a_i and hence $\mathbf{x}$ when

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{f} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

in the cases (i) $\mu = 2$ and (ii) $\mu = 1$.

8C Vectors and Matrices

Prove that the eigenvalues of a Hermitian matrix are real and that eigenvectors corresponding to distinct eigenvalues are orthogonal (i.e. $\mathbf{e}_i^* \cdot \mathbf{e}_j = 0$).

Let A be a real 3×3 non-zero antisymmetric matrix. Show that iA is Hermitian. Hence show that there exists a (complex) eigenvector $\mathbf{e}_1$ such $A\mathbf{e}_1 = \lambda \mathbf{e}_1$, where λ is imaginary.

Show further that there exist real vectors $\mathbf{u}$ and $\mathbf{v}$ and a real number θ such that

$$A\mathbf{u} = \theta\mathbf{v} \qquad \text{and} \qquad A\mathbf{v} = -\theta\mathbf{u}.$$

Show also that A has a real eigenvector $\mathbf{e}_3$ such that $A\mathbf{e}_3 = 0$.

Let $R = I + \displaystyle\sum_{n=1}^{\infty} \frac{A^n}{n!}$. By considering the action of R on $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{e_3}$, show that R is a rotation matrix.

9F Analysis I

Investigate the convergence of the series

(i) $$\sum_{n=2}^{\infty} \frac{1}{n^p (\log n)^q}$$

(ii) $$\sum_{n=3}^{\infty} \frac{1}{n\,(\log \log n)^r}$$

for positive real values of p, q and r.

[*You may assume that for any positive real value of* α, $\log n < n^{\alpha}$ *for* n *sufficiently large. You may assume standard tests for convergence, provided that they are clearly stated.*]

10D Analysis I

(a) State and prove the intermediate value theorem.

(b) An *interval* is a subset I of $\mathbb{R}$ with the property that if x and y belong to I and $x < z < y$ then z also belongs to I. Prove that if I is an interval and f is a continuous function from I to $\mathbb{R}$ then $f(I)$ is an interval.

(c) For each of the following three pairs (I, J) of intervals, either exhibit a continuous function f from I to $\mathbb{R}$ such that $f(I) = J$ or explain briefly why no such continuous function exists:

(i) $I = [0,1]$, $\quad J = [0,\infty)$;

(ii) $I = (0,1]$, $\quad J = [0,\infty)$;

(iii) $I = (0,1]$, $\quad J = (-\infty,\infty)$.

11D Analysis I

(a) Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ and suppose that both f and g are differentiable at the real number x. Prove that the product fg is also differentiable at x.

(b) Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let $g(x) = x^2 f(x)$ for every x. Prove that g is differentiable at x if and only if either $x = 0$ or f is differentiable at x.

(c) Now let f be any continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let $g(x) = f(x)^2$ for every x. Prove that g is differentiable at x if and only if at least one of the following two possibilities occurs:

(i) f is differentiable at x;

(ii) $f(x) = 0$ and

$$\frac{f(x+h)}{|h|^{1/2}} \longrightarrow 0 \quad \text{as} \quad h \to 0 .$$

12E Analysis I

Let $\sum_{n=0}^{\infty} a_n z^n$ be a complex power series. Prove that there exists an $R \in [0, \infty]$ such that the series converges for every z with $|z| < R$ and diverges for every z with $|z| > R$.

Find the value of R for each of the following power series:

(i) $$\sum_{n=1}^{\infty} \frac{1}{n^2} z^n ;$$

(ii) $$\sum_{n=0}^{\infty} z^{n!} .$$

In each case, determine at which points on the circle $|z| = R$ the series converges.

END OF PAPER