UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2007

List of Courses

Paper 1, Section I

1A Algebra and Geometry

(i) The spherical polar unit basis vectors $\mathbf{e}_r, \mathbf{e}_\phi$ and $\mathbf{e}_\theta$ in $\mathbb{R}^3$ are given in terms of the Cartesian unit basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ by

$$\begin{aligned}\mathbf{e}_r &= \mathbf{i}\cos\phi\sin\theta + \mathbf{j}\sin\phi\sin\theta + \mathbf{k}\cos\theta\,,\\ \mathbf{e}_\theta &= \mathbf{i}\cos\phi\cos\theta + \mathbf{j}\sin\phi\cos\theta - \mathbf{k}\sin\theta\,,\\ \mathbf{e}_\phi &= -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi\,.\end{aligned}$$

Express $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ in terms of $\mathbf{e}_r, \mathbf{e}_\phi$ and $\mathbf{e}_\theta$.

(ii) Use suffix notation to prove the following identity for the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in $\mathbb{R}^3$:

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\,\mathbf{A}\,.$$

Paper 1, Section I

2B Algebra and Geometry

For the equations

$$\begin{aligned}px + y + z &= 1,\\ x + 2y + 4z &= t,\\ x + 4y + 10z &= t^2,\end{aligned}$$

find the values of p and t for which

(i) there is a unique solution;

(ii) there are infinitely many solutions;

(iii) there is no solution.

Paper 1, Section II

5C Algebra and Geometry

(i) Describe geometrically the following surfaces in three-dimensional space:

(a) $\mathbf{r}\cdot\mathbf{u} = \alpha|\mathbf{r}|$, where $0 < |\alpha| < 1$;

(b) $|\mathbf{r} - (\mathbf{r}\cdot\mathbf{u})\mathbf{u}| = \beta$, where $\beta > 0$.

Here α and β are fixed scalars and $\mathbf{u}$ is a fixed unit vector. You should identify the meaning of α, β and $\mathbf{u}$ for these surfaces.

(ii) The plane $\mathbf{n}\cdot\mathbf{r} = p$, where $\mathbf{n}$ is a fixed unit vector, and the sphere with centre $\mathbf{c}$ and radius a intersect in a circle with centre $\mathbf{b}$ and radius ρ.

(a) Show that $\mathbf{b} - \mathbf{c} = \lambda\mathbf{n}$, where you should give λ in terms of a and ρ.

(b) Find ρ in terms of $\mathbf{c}, \mathbf{n}, a$ and p.

Paper 1, Section II

6C Algebra and Geometry

Let $\mathcal{M} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$\mathbf{x} \mapsto \mathbf{x}' = a\mathbf{x} + b(\mathbf{n} \times \mathbf{x})\,,$$

where a and b are positive scalar constants, and $\mathbf{n}$ is a unit vector.

(i) By considering the effect of $\mathcal{M}$ on $\mathbf{n}$ and on a vector orthogonal to $\mathbf{n}$, describe geometrically the action of $\mathcal{M}$.

(ii) Express the map $\mathcal{M}$ as a matrix M using suffix notation. Find a, b and $\mathbf{n}$ in the case

$$\mathrm{M} = \begin{pmatrix} 2 & -2 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & 2 \end{pmatrix}.$$

(iii) Find, in the general case, the inverse map (i.e. express $\mathbf{x}$ in terms of $\mathbf{x}'$ in vector form).

Paper 1, Section II

7C Algebra and Geometry

Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in a real vector space with scalar product denoted by $\mathbf{x} \cdot \mathbf{y}$. Prove that $(\mathbf{x} \cdot \mathbf{y})^{\mathbf{2}} \leqslant (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$, and prove also that $(\mathbf{x} \cdot \mathbf{y})^{\mathbf{2}} = (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$ if and only if $\mathbf{x} = \lambda\mathbf{y}$ for some scalar λ.

(i) By considering suitable vectors in $\mathbb{R}^3$, or otherwise, prove that the inequality $x^2 + y^2 + z^2 \geqslant yz + zx + xy$ holds for any real numbers x, y and z.

(ii) By considering suitable vectors in $\mathbb{R}^4$, or otherwise, show that only one choice of real numbers x, y, z satisfies $3(x^2 + y^2 + z^2 + 4) - 2(yz + zx + xy) - 4(x + y + z) = 0$, and find these numbers.

Paper 1, Section II

8A Algebra and Geometry

(i) Show that any line in the complex plane $\mathbb{C}$ can be represented in the form

$$\bar{c}z + c\bar{z} + r = 0\,,$$

where $c \in \mathbb{C}$ and $r \in \mathbb{R}$.

(ii) If z and u are two complex numbers for which

$$\left|\frac{z+u}{z+\bar{u}}\right| = 1\,,$$

show that either z or u is real.

(iii) Show that any Möbius transformation

$$w = \frac{az+b}{cz+d} \quad (bc - ad \neq 0)$$

that maps the real axis $z = \bar{z}$ into the unit circle $|w| = 1$ can be expressed in the form

$$w = \lambda \frac{z+k}{z+\bar{k}}\,,$$

where $\lambda, k \in \mathbb{C}$ and $|\lambda| = 1$.

Paper 3, Section I

1D Algebra and Geometry

Prove that every permutation of $\{1, \ldots, n\}$ may be expressed as a product of disjoint cycles.

Let $\sigma = (1234)$ and let $\tau = (345)(678)$. Write $\sigma\tau$ as a product of disjoint cycles. What is the order of $\sigma\tau$?

Paper 3, Section I

2D Algebra and Geometry

What does it mean to say that groups G and H are *isomorphic*?

Prove that no two of C_8, $C_4 \times C_2$ and $C_2 \times C_2 \times C_2$ are isomorphic. [Here C_n denotes the cyclic group of order n.]

Give, with justification, a group of order 8 that is not isomorphic to any of those three groups.

Paper 3, Section II

5D Algebra and Geometry

Let x be an element of a finite group G. What is meant by the *order* of x? Prove that the order of x must divide the order of G. [*No version of Lagrange's theorem or the Orbit-Stabilizer theorem may be used without proof.*]

If G is a group of order n, and d is a divisor of n with $d < n$, is it always true that G must contain an element of order d? Justify your answer.

Prove that if m and n are coprime then the group $C_m \times C_n$ is cyclic.

If m and n are not coprime, can it happen that $C_m \times C_n$ is cyclic?

[Here C_n denotes the cyclic group of order n.]

Paper 3, Section II

6D Algebra and Geometry

What does it mean to say that a subgroup H of a group G is *normal*? Give, with justification, an example of a subgroup of a group that is normal, and also an example of a subgroup of a group that is not normal.

If H is a normal subgroup of G, explain carefully how to make the set of (left) cosets of H into a group.

Let H be a normal subgroup of a finite group G. Which of the following are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) If G is cyclic then H and G/H are cyclic.

(ii) If H and G/H are cyclic then G is cyclic.

(iii) If G is abelian then H and G/H are abelian.

(iv) If H and G/H are abelian then G is abelian.

Paper 3, Section II

7D Algebra and Geometry

Let A be a real symmetric $n \times n$ matrix. Prove that every eigenvalue of A is real, and that eigenvectors corresponding to distinct eigenvalues are orthogonal. Indicate clearly where in your argument you have used the fact that A is real.

What does it mean to say that a real $n \times n$ matrix P is *orthogonal*? Show that if P is orthogonal and A is as above then $P^{-1}AP$ is symmetric. If P is any real invertible matrix, must $P^{-1}AP$ be symmetric? Justify your answer.

Give, with justification, real 2×2 matrices B, C, D, E with the following properties:

(i) B has no real eigenvalues;

(ii) C is not diagonalisable over $\mathbb{C}$;

(iii) D is diagonalisable over $\mathbb{C}$, but not over $\mathbb{R}$;

(iv) E is diagonalisable over $\mathbb{R}$, but does not have an orthonormal basis of eigenvectors.

Paper 3, Section II

8D Algebra and Geometry

In the group of Möbius maps, what is the order of the Möbius map $z \mapsto \frac{1}{z}$? What is the order of the Möbius map $z \mapsto \frac{1}{1-z}$?

Prove that every Möbius map is conjugate either to a map of the form $z \mapsto \mu z$ (some $\mu \in \mathbb{C}$) or to the map $z \mapsto z+1$. Is $z \mapsto z+1$ conjugate to a map of the form $z \mapsto \mu z$?

Let f be a Möbius map of order n, for some positive integer n. Under the action on $\mathbb{C} \cup \{\infty\}$ of the group generated by f, what are the various sizes of the orbits? Justify your answer.

Paper 1, Section I

3F Analysis

Prove that, for positive real numbers a and b,

$$2\sqrt{ab} \leqslant a+b\,.$$

For positive real numbers $a_1, a_2, \ldots$, prove that the convergence of

$$\sum_{n=1}^{\infty} a_n$$

implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}\,.$$

Paper 1, Section I

4D Analysis

Let $\sum_{n=0}^{\infty} a_n z^n$ be a complex power series. Show that there exists $R \in [0,\infty]$ such that $\sum_{n=0}^{\infty} a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$.

Find the value of R for the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}\,.$$

Paper 1, Section II

9F Analysis

Let $a_1 = \sqrt{2}$, and consider the sequence of positive real numbers defined by

$$a_{n+1} = \sqrt{2+\sqrt{a_n}}\,, \qquad n = 1,2,3,\ldots\,.$$

Show that $a_n \leqslant 2$ for all n. Prove that the sequence $a_1, a_2, \ldots$ converges to a limit.

Suppose instead that $a_1 = 4$. Prove that again the sequence $a_1, a_2, \ldots$ converges to a limit.

Prove that the limits obtained in the two cases are equal.

Paper 1, Section II

10E Analysis

State and prove the Mean Value Theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that, for every $x \in \mathbb{R}$, $f''(x)$ exists and is non-negative.

(i) Show that if $x \leqslant y$ then $f'(x) \leqslant f'(y)$.

(ii) Let $\lambda \in (0,1)$ and $a < b$. Show that there exist x and y such that

$$f\big(\lambda a + (1-\lambda)b\big) = f(a) + (1-\lambda)(b-a)f'(x) = f(b) - \lambda(b-a)f'(y)$$

and that

$$f\big(\lambda a + (1-\lambda)b\big) \leqslant \lambda f(a) + (1-\lambda)f(b)\,.$$

Paper 1, Section II

11E Analysis

Let $a < b$ be real numbers, and let $f : [a,b] \to \mathbb{R}$ be continuous. Show that f is bounded on $[a,b]$, and that there exist $c, d \in [a,b]$ such that for all $x \in [a,b]$, $f(c) \leqslant f(x) \leqslant f(d)$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to+\infty} g(x) = \lim_{x\to-\infty} g(x) = 0\,.$$

Show that g is bounded. Show also that, if a and c are real numbers with $0 < c \leqslant g(a)$, then there exists $x \in \mathbb{R}$ with $g(x) = c$.

Paper 1, Section II

12D Analysis

Explain carefully what it means to say that a bounded function $f:\ [0,1] \to \mathbb{R}$ is *Riemann integrable.*

Prove that every continuous function $f:\ [0,1] \to \mathbb{R}$ is Riemann integrable.

For each of the following functions from $[0,1]$ to $\mathbb{R}$, determine with proof whether or not it is Riemann integrable:

(i) the function $f(x) = x \sin \dfrac{1}{x}$ for $x \neq 0$, with $f(0) = 0$;

(ii) the function $g(x) = \sin \dfrac{1}{x}$ for $x \neq 0$, with $g(0) = 0$.

Paper 2, Section I

1B Differential Equations

Find the solution $y(x)$ of the equation

$$y'' - 6y' + 9y = \cos(2x)\,\mathrm{e}^{3x}$$

that satisfies $y(0) = 0$ and $y'(0) = 1$.

Paper 2, Section I

2B Differential Equations

Investigate the stability of:

(i) the equilibrium points of the equation

$$\frac{dy}{dt} = (y^2 - 4)\tan^{-1}(y)\,;$$

(ii) the constant solutions ($u_{n+1} = u_n$) of the discrete equation

$$u_{n+1} = \frac{1}{2}u_n^2(1 + u_n)\,.$$

Paper 2, Section II

5B Differential Equations

(i) The function $y(z)$ satisfies the equation

$$y'' + p(z)y' + q(z)y = 0\,.$$

Give the definitions of the terms *ordinary point, singular point,* and *regular singular point* for this equation.

(ii) For the equation

$$4zy'' + 2y' + y = 0\,,$$

classify the point $z = 0$ according to the definitions you gave in (i), and find the series solutions about $z = 0$. Identify these solutions in closed form.

Paper 2, Section II

6B Differential Equations

Find the most general solution of the equation

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 1$$

by making the change of variables

$$\xi = x + 2y, \qquad \eta = x + 3y\,.$$

Find the solution that satisfies $u = 0$ and $\partial u/\partial y = x$ when $y = 0$.

Paper 2, Section II

7B Differential Equations

(i) Find, in the form of an integral, the solution of the equation

$$\alpha\frac{dy}{dt} + y = f(t)$$

that satisfies $y \to 0$ as $t \to -\infty$. Here $f(t)$ is a general function and α is a positive constant.

Hence find the solution in each of the cases:

(a) $f(t) = \delta(t)$;

(b) $f(t) = H(t)$, where $H(t)$ is the Heaviside step function.

(ii) Find and sketch the solution of the equation

$$\frac{dy}{dt} + y = H(t) - H(t-1)\,,$$

given that $y(0) = 0$ and $y(t)$ is continuous.

Paper 2, Section II

8B Differential Equations

(i) Find the general solution of the difference equation

$$u_{k+1} + 5u_k + 6u_{k-1} = 12k + 1\,.$$

(ii) Find the solution of the equation

$$y_{k+1} + 5y_k + 6y_{k-1} = 2^k$$

that satisfies $y_0 = y_1 = 1$. Hence show that, for any positive integer n, the quantity $2^n - 26(-3)^n$ is divisible by 10.

Paper 4, Section I

3C Dynamics

A rocket, moving vertically upwards, ejects gas vertically downwards at speed u relative to the rocket. Derive, giving careful explanations, the equation of motion

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - gm\,,$$

where v and m are the speed and total mass of the rocket (including fuel) at time t.

If u is constant and the rocket starts from rest with total mass m_0, show that

$$m = m_0 e^{-(gt+v)/u}\,.$$

Paper 4, Section I

4C Dynamics

Sketch the graph of $y = 3x^2 - 2x^3$.

A particle of unit mass moves along the x axis in the potential $V(x) = 3x^2 - 2x^3$. Sketch the phase plane, and describe briefly the motion of the particle on the different trajectories.

Paper 4, Section II

9C Dynamics

A small ring of mass m is threaded on a smooth rigid wire in the shape of a parabola given by $x^2 = 4az$, where x measures horizontal distance and z measures distance vertically upwards. The ring is held at height $z = h$, then released.

(i) Show by dimensional analysis that the period of oscillations, T, can be written in the form

$$T = (a/g)^{1/2} G(h/a)$$

for some function G.

(ii) Show that G is given by

$$G(\beta) = 2\sqrt{2}\int_{-1}^{1}\left(\frac{1+\beta u^2}{1-u^2}\right)^{\frac{1}{2}} du\,,$$

and find, to first order in h/a, the period of small oscillations.

Paper 4, Section II

10C Dynamics

A particle of mass m experiences, at the point with position vector $\mathbf{r}$, a force $\mathbf{F}$ given by

$$\mathbf{F} = -k\mathbf{r} - e\dot{\mathbf{r}} \times \mathbf{B}\,,$$

where k and e are positive constants and $\mathbf{B}$ is a constant, uniform, vector field.

(i) Show that $m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + k\mathbf{r} \cdot \mathbf{r}$ is constant. Give a physical interpretation of each term and a physical explanation of the fact that $\mathbf{B}$ does not arise in this expression.

(ii) Show that $m(\dot{\mathbf{r}} \times \mathbf{r}) \cdot \mathbf{B} + \frac{1}{2}e(\mathbf{r} \times \mathbf{B}) \cdot (\mathbf{r} \times \mathbf{B})$ is constant.

(iii) Given that the particle was initially at rest at $\mathbf{r}_0$, derive an expression for $\mathbf{r} \cdot \mathbf{B}$ at time t.

Paper 4, Section II

11C Dynamics

A particle moves in the gravitational field of the Sun. The angular momentum per unit mass of the particle is h and the mass of the Sun is M. Assuming that the particle moves in a plane, write down the equations of motion in polar coordinates, and derive the equation

$$\frac{d^2u}{d\theta^2} + u = k\,,$$

where $u = 1/r$ and $k = GM/h^2$.

Write down the equation of the orbit (u as a function of θ), given that the particle moves with the escape velocity and is at the perihelion of its orbit, a distance r_0 from the Sun, when $\theta = 0$. Show that

$$\sec^4(\theta/2)\frac{d\theta}{dt} = \frac{h}{r_0^2}$$

and hence that the particle reaches a distance $2r_0$ from the Sun at time $8r_0^2/(3h)$.

Paper 4, Section II

12C Dynamics

The ith particle of a system of N particles has mass m_i and, at time t, position vector $\mathbf{r}_i$ with respect to an origin O. It experiences an external force $\mathbf{F}_i^e$, and also an internal force $\mathbf{F}_{ij}$ due to the jth particle (for each $j = 1, ..., N$, $j \neq i$), where $\mathbf{F}_{ij}$ is parallel to $\mathbf{r}_i - \mathbf{r}_j$ and Newton's third law applies.

(i) Show that the position of the centre of mass, $\mathbf{X}$, satisfies

$$M \frac{d^2\mathbf{X}}{dt^2} = \mathbf{F}^e \,,$$

where M is the total mass of the system and $\mathbf{F}^e$ is the sum of the external forces.

(ii) Show that the total angular momentum of the system about the origin, $\mathbf{L}$, satisfies

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} \,,$$

where $\mathbf{N}$ is the total moment about the origin of the external forces.

(iii) Show that $\mathbf{L}$ can be expressed in the form

$$\mathbf{L} = M\mathbf{X} \times \mathbf{V} + \sum_i m_i \mathbf{r}_i' \times \mathbf{v}_i' \,,$$

where $\mathbf{V}$ is the velocity of the centre of mass, $\mathbf{r}_i'$ is the position vector of the ith particle relative to the centre of mass, and $\mathbf{v}_i'$ is the velocity of the ith particle relative to the centre of mass.

(iv) In the case $N = 2$ when the internal forces are derived from a potential $U(|\mathbf{r}|)$, where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and there are no external forces, show that

$$\frac{dT}{dt} + \frac{dU}{dt} = 0 \,,$$

where T is the total kinetic energy of the system.

Paper 4, Section I

1E Numbers and Sets

(i) Use Euclid's algorithm to find all pairs of integers x and y such that

$$7x + 18y = 1\,.$$

(ii) Show that, if n is odd, then $n^3 - n$ is divisible by 24.

Paper 4, Section I

2E Numbers and Sets

For integers k and n with $0 \leqslant k \leqslant n$, define $\binom{n}{k}$. Arguing from your definition, show that

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

for all integers k and n with $1 \leqslant k \leqslant n-1$.

Use induction on k to prove that

$$\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$$

for all non-negative integers k and n.

Paper 4, Section II

5E Numbers and Sets

State and prove the Inclusion–Exclusion principle.

The keypad on a cash dispenser is broken. To withdraw money, a customer is required to key in a 4-digit number. However, the key numbered 0 will only function if either the immediately preceding two keypresses were both 1, or the very first key pressed was 2. Explaining your reasoning clearly, use the Inclusion–Exclusion Principle to find the number of 4-digit codes which can be entered.

Paper 4, Section II

6E Numbers and Sets

Stating carefully any results about countability you use, show that for any $d \geqslant 1$ the set $\mathbb{Z}[X_1, \ldots, X_d]$ of polynomials with integer coefficients in d variables is countable. By taking $d = 1$, deduce that there exist uncountably many transcendental numbers.

Show that there exists a sequence $x_1, x_2, \ldots$ of real numbers with the property that $f(x_1, \ldots, x_d) \neq 0$ for every $d \geqslant 1$ and for every non-zero polynomial $f \in \mathbb{Z}[X_1, \ldots, X_d]$.

[*You may assume without proof that* $\mathbb{R}$ *is uncountable.*]

Paper 4, Section II

7E Numbers and Sets

Let x_n $(n = 1, 2, \dots)$ be real numbers.

What does it mean to say that the sequence $(x_n)_{n=1}^{\infty}$ converges?

What does it mean to say that the series $\sum_{n=1}^{\infty} x_n$ converges?

Show that if $\sum_{n=1}^{\infty} x_n$ is convergent, then $x_n \to 0$. Show that the converse can be false.

Sequences of positive real numbers x_n, y_n $(n \geqslant 1)$ are given, such that the inequality

$$y_{n+1} \leqslant y_n - \frac{1}{2}\min(x_n, y_n)$$

holds for all $n \geqslant 1$. Show that, if $\sum_{n=1}^{\infty} x_n$ diverges, then $y_n \to 0$.

Paper 4, Section II

8E Numbers and Sets

(i) Let p be a prime number, and let x and y be integers such that p divides xy. Show that at least one of x and y is divisible by p. Explain how this enables one to prove the Fundamental Theorem of Arithmetic.

[*Standard properties of highest common factors may be assumed without proof.*]

(ii) State and prove the Fermat-Euler Theorem.

Let $1/359$ have decimal expansion $0 \cdot a_1 a_2 \dots$ with $a_n \in \{0, 1, \dots, 9\}$. Use the fact that $60^2 \equiv 10 \pmod{359}$ to show that, for every n, $a_n = a_{n+179}$.

Paper 2, Section I

3F Probability

Let X and Y be independent random variables, each uniformly distributed on $[0, 1]$. Let $U = \min(X, Y)$ and $V = \max(X, Y)$. Show that $\mathbb{E}U = \frac{1}{3}$, and hence find the covariance of U and V.

Paper 2, Section I

4F Probability

Let X be a normally distributed random variable with mean 0 and variance 1. Define, and determine, the moment generating function of X. Compute $\mathbb{E}X^r$ for $r = 0, 1, 2, 3, 4$.

Let Y be a normally distributed random variable with mean μ and variance σ^2. Determine the moment generating function of Y.

Paper 2, Section II

9F Probability

Let N be a non-negative integer-valued random variable with

$$P\{N = r\} = p_r, \quad r = 0, 1, 2, \ldots$$

Define $\mathbb{E}N$, and show that

$$\mathbb{E}N = \sum_{n=1}^{\infty} P\{N \geqslant n\}\,.$$

Let X_1, X_2, $\ldots$ be a sequence of independent and identically distributed continuous random variables. Let the random variable N mark the point at which the sequence stops decreasing: that is, $N \geqslant 2$ is such that

$$X_1 \geqslant X_2 \geqslant \ldots \geqslant X_{N-1} < X_N\,,$$

where, if there is no such finite value of N, we set $N = \infty$. Compute $P\{N = r\}$, and show that $P\{N = \infty\} = 0$. Determine $\mathbb{E}N$.

Paper 2, Section II

10F Probability

Let X and Y be independent non-negative random variables, with densities f and g respectively. Find the joint density of $U = X$ and $V = X + aY$, where a is a positive constant.

Let X and Y be independent and exponentially distributed random variables, each with density

$$f(x) = \lambda e^{-\lambda x}, \quad x \geqslant 0\,.$$

Find the density of $X + \frac{1}{2}Y$. Is it the same as the density of the random variable $\max(X, Y)$?

Paper 2, Section II

11F Probability

Let $A_1, A_2, \ldots, A_n$ $(n \geqslant 2)$ be events in a sample space. For each of the following statements, either prove the statement or provide a counterexample.

(i)

$$P\left(\bigcap_{k=2}^{n} A_k \,\middle|\, A_1\right) = \prod_{k=2}^{n} P\left(A_k \,\middle|\, \bigcap_{r=1}^{k-1} A_r\right), \quad \text{provided} \quad P\left(\bigcap_{k=1}^{n-1} A_k\right) > 0\,.$$

(ii)

$$\text{If} \quad \sum_{k=1}^{n} P(A_k) > n-1 \quad \text{then} \quad P\left(\bigcap_{k=1}^{n} A_k\right) > 0\,.$$

(iii)

$$\text{If} \quad \sum_{i<j} P\left(A_i \bigcap A_j\right) > \binom{n}{2} - 1 \quad \text{then} \quad P\left(\bigcap_{k=1}^{n} A_k\right) > 0\,.$$

(iv) If B is an event and if, for each k, $\{B, A_k\}$ is a pair of independent events, then $\{B, \cup_{k=1}^{n} A_k\}$ is also a pair of independent events.

Paper 2, Section II

12F Probability

Let A, B and C be three random points on a sphere with centre O. The positions of A, B and C are independent, and each is uniformly distributed over the surface of the sphere. Calculate the probability density function of the angle $\angle AOB$ formed by the lines OA and OB.

Calculate the probability that all three of the angles $\angle AOB$, $\angle AOC$ and $\angle BOC$ are acute. [**Hint:** *Condition on the value of the angle $\angle AOB$.*]

Paper 3, Section I

3A Vector Calculus

(i) Give definitions for the unit tangent vector $\hat{\mathbf{T}}$ and the curvature κ of a parametrised curve $\mathbf{x}(t)$ in $\mathbb{R}^3$. Calculate $\hat{\mathbf{T}}$ and κ for the circular helix

$$\mathbf{x}(t) = (a\cos t\,,\; a\sin t\,,\; bt)\,,$$

where a and b are constants.

(ii) Find the normal vector and the equation of the tangent plane to the surface S in $\mathbb{R}^3$ given by

$$z = x^2y^3 - y + 1$$

at the point $x = 1$, $y = 1$, $z = 1$.

Paper 3, Section I

4A Vector Calculus

By using suffix notation, prove the following identities for the vector fields $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^3$:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})\,;$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})\,.$$

Paper 3, Section II

9A Vector Calculus

(i) Define what is meant by a conservative vector field. Given a vector field $\mathbf{A} = (A_1(x,y), A_2(x,y))$ and a function $\psi(x,y)$ defined in $\mathbb{R}^2$, show that, if $\psi\mathbf{A}$ is a conservative vector field, then

$$\psi\left(\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x}\right) = A_2\frac{\partial \psi}{\partial x} - A_1\frac{\partial \psi}{\partial y}\,.$$

(ii) Given two functions $P(x,y)$ and $Q(x,y)$ defined in $\mathbb{R}^2$, prove Green's theorem,

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\,,$$

where C is a simple closed curve bounding a region R in $\mathbb{R}^2$.

Through an appropriate choice for P and Q, find an expression for the area of the region R, and apply this to evaluate the area of the ellipse bounded by the curve

$$x = a\cos\theta\,, \qquad y = b\sin\theta\,, \qquad 0 \leqslant \theta \leqslant 2\pi\,.$$

Paper 3, Section II

10A Vector Calculus

For a given charge distribution $\rho(x,y,z)$ and divergence-free current distribution $\mathbf{J}(x,y,z)$ (i.e. $\nabla\cdot\mathbf{J}=0$) in $\mathbb{R}^3$, the electric and magnetic fields $\mathbf{E}(x,y,z)$ and $\mathbf{B}(x,y,z)$ satisfy the equations

$$\nabla\times\mathbf{E}=0\,,\qquad \nabla\cdot\mathbf{B}=0\,,\qquad \nabla\cdot\mathbf{E}=\rho\,,\qquad \nabla\times\mathbf{B}=\mathbf{J}\,.$$

The radiation flux vector $\mathbf{P}$ is defined by $\mathbf{P}=\mathbf{E}\times\mathbf{B}$. For a closed surface S around a region V, show using Gauss' theorem that the flux of the vector $\mathbf{P}$ through S can be expressed as

$$\iint_S \mathbf{P}\cdot\mathbf{dS} = -\iiint_V \mathbf{E}\cdot\mathbf{J}\,dV\,. \tag{$*$}$$

For electric and magnetic fields given by

$$\mathbf{E}(x,y,z)=(z,\ 0,\ x)\,,\qquad \mathbf{B}(x,y,z)=(0,\ -xy,\ xz)\,,$$

find the radiation flux through the quadrant of the unit spherical shell given by

$$x^2+y^2+z^2=1\,,\quad \text{with}\quad 0\leqslant x\leqslant 1\,,\quad 0\leqslant y\leqslant 1\,,\quad -1\leqslant z\leqslant 1\,.$$

[*If you use* ($*$), *note that an open surface has been specified.*]

Paper 3, Section II

11A Vector Calculus

The function $\phi(x,y,z)$ satisfies $\nabla^2\phi=0$ in V and $\phi=0$ on S, where V is a region of $\mathbb{R}^3$ which is bounded by the surface S. Prove that $\phi=0$ everywhere in V.

Deduce that there is at most one function $\psi(x,y,z)$ satisfying $\nabla^2\psi=\rho$ in V and $\psi=f$ on S, where $\rho(x,y,z)$ and $f(x,y,z)$ are given functions.

Given that the function $\psi=\psi(r)$ depends only on the radial coordinate $r=|\mathbf{x}|$, use Cartesian coordinates to show that

$$\nabla\psi=\frac{1}{r}\frac{d\psi}{dr}\mathbf{x}\,,\qquad \nabla^2\psi=\frac{1}{r}\frac{d^2(r\psi)}{dr^2}\,.$$

Find the general solution in this radial case for $\nabla^2\psi=c$ where c is a constant.

Find solutions $\psi(r)$ for a solid sphere of radius $r=2$ with a central cavity of radius $r=1$ in the following three regions:

(i) $0\leqslant r\leqslant 1$ where $\nabla^2\psi=0$ and $\psi(1)=1$ and ψ bounded as $r\to 0$;

(ii) $1\leqslant r\leqslant 2$ where $\nabla^2\psi=1$ and $\psi(1)=\psi(2)=1$;

(iii) $r\geqslant 2$ where $\nabla^2\psi=0$ and $\psi(2)=1$ and $\psi\to 0$ as $r\to\infty$.

Paper 3, Section II

12A Vector Calculus

Show that any second rank Cartesian tensor P_{ij} in $\mathbb{R}^3$ can be written as a sum of a symmetric tensor and an antisymmetric tensor. Further, show that P_{ij} can be decomposed into the following terms

$$P_{ij} = P\delta_{ij} + S_{ij} + \epsilon_{ijk}A_k\,, \qquad (\dagger)$$

where S_{ij} is symmetric and traceless. Give expressions for P, S_{ij} and A_k explicitly in terms of P_{ij}.

For an isotropic material, the stress P_{ij} can be related to the strain T_{ij} through the stress–strain relation, $P_{ij} = c_{ijkl}\,T_{kl}$, where the elasticity tensor is given by

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

and α, β and γ are scalars. As in ($\dagger$), the strain T_{ij} can be decomposed into its trace T, a symmetric traceless tensor W_{ij} and a vector V_k. Use the stress–strain relation to express each of T, W_{ij} and V_k in terms of P, S_{ij} and A_k.

Hence, or otherwise, show that if T_{ij} is symmetric then so is P_{ij}. Show also that the stress-strain relation can be written in the form

$$P_{ij} = \lambda\,\delta_{ij}T_{kk} + \mu\,T_{ij}\,,$$

where μ and λ are scalars.