UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 8 June 2007 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

Suppose that $\alpha : V \to W$ is a linear map of finite-dimensional complex vector spaces. What is the dual map α^* of the dual vector spaces?

Suppose that we choose bases of V, W and take the corresponding dual bases of the dual vector spaces. What is the relation between the matrices that represent α and α^* with respect to these bases? Justify your answer.

2G Groups, Rings and Modules

If p is a prime, how many abelian groups of order p^4 are there, up to isomorphism?

3H Analysis II

Define uniform convergence for a sequence $f_1, f_2, \ldots$ of real-valued functions on the interval $(0,1)$.

For each of the following sequences of functions on $(0,1)$, find the pointwise limit function. Which of these sequences converge uniformly on $(0,1)$?

(i) $f_n(x) = \log\,(x + \frac{1}{n})$,

(ii) $f_n(x) = \cos\,(\frac{x}{n})$.

Justify your answers.

4H Complex Analysis

State the argument principle.

Show that if f is an analytic function on an open set $U \subset \mathbb{C}$ which is one-to-one, then $f'(z) \neq 0$ for all $z \in U$.

5B Methods

Show that the general solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \,,$$

where c is a constant, is

$$y = f(x + ct) + g(x - ct) \,,$$

where f and g are twice differentiable functions. Briefly discuss the physical interpretation of this solution.

Calculate $y(x,t)$ subject to the initial conditions

$$y(x,0) = 0 \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = \psi(x) \,.$$

6B Quantum Mechanics

A particle moving in one space dimension with wave-function $\Psi(x,t)$ obeys the time-dependent Schrödinger equation. Write down the probability density, ρ, and current density, j, in terms of the wave-function and show that they obey the equation

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \,.$$

The wave-function is

$$\Psi(x,t) = \left(e^{ikx} + R\, e^{-ikx}\right) e^{-iEt/\hbar} \,,$$

where $E = \hbar^2 k^2 / 2m$ and R is a constant, which may be complex. Evaluate j.

7E Electromagnetism

Write down Faraday's law of electromagnetic induction for a moving circuit $C(t)$ in a magnetic field $\mathbf{B}(\mathbf{x},t)$. Explain carefully the meaning of each term in the equation.

A thin wire is bent into a circular loop of radius a. The loop lies in the (x,z)-plane at time $t = 0$. It spins steadily with angular velocity $\Omega\mathbf{k}$, where Ω is a constant and $\mathbf{k}$ is a unit vector in the z-direction. A spatially uniform magnetic field $\mathbf{B} = B_0(\cos\omega t, \sin\omega t, 0)$ is applied, with B_0 and ω both constant. If the resistance of the wire is R, find the current in the wire at time t.

8F Numerical Analysis

Given $f \in C^3[0,2]$, we approximate $f'(0)$ by the linear combination

$$\mu(f) = -\frac{3}{2}f(0) + 2f(1) - \frac{1}{2}f(2)\,.$$

Using the Peano kernel theorem, determine the least constant c in the inequality

$$|f'(0) - \mu(f)| \le c\,\|f'''\|_\infty\,,$$

and give an example of f for which the inequality turns into equality.

9C Markov Chains

For a Markov chain with state space S, define what is meant by the following:

(i) states $i, j \in S$ *communicate*;

(ii) state $i \in S$ is *recurrent*.

Prove that communication is an equivalence relation on S and that if two states i, j communicate and i is recurrent then j is recurrent.

SECTION II

10G Linear Algebra

(i) State and prove the Cayley–Hamilton theorem for square complex matrices.

(ii) A square matrix A is *of order* n for a strictly positive integer n if $A^n = I$ and no smaller positive power of A is equal to I.

Determine the order of a complex 2×2 matrix A of trace zero and determinant 1.

11G Groups, Rings and Modules

A regular icosahedron has 20 faces, 12 vertices and 30 edges. The group G of its rotations acts transitively on the set of faces, on the set of vertices and on the set of edges.

(i) List the conjugacy classes in G and give the size of each.

(ii) Find the order of G and list its normal subgroups.

[*A normal subgroup of* G *is a union of conjugacy classes in* G*.*]

12A Geometry

Write down the Riemannian metric for the upper half-plane model $\mathbf{H}$ of the hyperbolic plane. Describe, without proof, the group of isometries of $\mathbf{H}$ and the hyperbolic lines (i.e. the geodesics) on $\mathbf{H}$.

Show that for any two hyperbolic lines ℓ_1, ℓ_2, there is an isometry of $\mathbf{H}$ which maps ℓ_1 onto ℓ_2.

Suppose that g is a composition of two reflections in hyperbolic lines which are ultraparallel (i.e. do not meet either in the hyperbolic plane or at its boundary). Show that g cannot be an element of finite order in the group of isometries of $\mathbf{H}$.

[*Existence of a common perpendicular to two ultraparallel hyperbolic lines may be assumed. You might like to choose carefully which hyperbolic line to consider as a common perpendicular.*]

13H Analysis II

State and prove the Contraction Mapping Theorem.

Find numbers a and b, with $a < 0 < b$, such that the mapping $T : C[a,b] \to C[a,b]$ defined by

$$T(f)(x) = 1 + \int_0^x 3t\, f(t)\, dt$$

is a contraction, in the sup norm on $C[a,b]$. Deduce that the differential equation

$$\frac{dy}{dx} = 3xy, \qquad \text{with } y = 1 \text{ when } x = 0,$$

has a unique solution in some interval containing 0.

14A Metric and Topological Spaces

(a) For a subset A of a topological space X, define the *closure* $cl(A)$ of A. Let $f : X \to Y$ be a map to a topological space Y. Prove that f is continuous if and only if $f(cl(A)) \subseteq cl(f(A))$, for each $A \subseteq X$.

(b) Let M be a metric space. A subset S of M is called *dense* in M if the closure of S is equal to M.

Prove that if a metric space M is compact then it has a countable subset which is dense in M.

15F Complex Methods

(i) Use the definition of the Laplace transform of $f(t)$:

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt\,,$$

to show that, for $f(t) = t^n$,

$$L\{f(t)\} = F(s) = \frac{n!}{s^{n+1}}\,, \qquad L\{e^{at} f(t)\} = F(s-a) = \frac{n!}{(s-a)^{n+1}}\,.$$

(ii) Use contour integration to find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s+1)^2}\,.$$

(iii) Verify the result in (ii) by using the results in (i) and the convolution theorem.

(iv) Use Laplace transforms to solve the differential equation

$$f^{(iv)}(t) + 2f'''(t) + f''(t) = 0,$$

subject to the initial conditions

$$f(0) = f'(0) = f''(0) = 0, \qquad f'''(0) = 1\,.$$

16E Methods

Write down the Euler-Lagrange equation for extrema of the functional

$$I = \int_a^b F\left(y, y'\right) dx \, .$$

Show that a first integral of this equation is given by

$$F - y' \frac{\partial F}{\partial y'} = C \, .$$

A road is built between two points A and B in the plane $z = 0$ whose polar coordinates are $r = a$, $\theta = 0$ and $r = a$, $\theta = \pi/2$ respectively. Owing to congestion, the traffic speed at points along the road is kr^2 with k a positive constant. If the equation describing the road is $r = r(\theta)$, obtain an integral expression for the total travel time T from A to B.

[*Arc length in polar coordinates is given by* $ds^2 = dr^2 + r^2 d\theta^2$.]

Calculate T for the circular road $r = a$.

Find the equation for the road that minimises T and determine this minimum value.

17B Special Relativity

(a) A moving π^0 particle of rest-mass m_π decays into two photons of zero rest-mass,

$$\pi^0 \to \gamma + \gamma \, .$$

Show that

$$\sin \frac{\theta}{2} = \frac{m_\pi c^2}{2\sqrt{E_1 E_2}} \, ,$$

where θ is the angle between the three-momenta of the two photons and E_1, E_2 are their energies.

(b) The π^- particle of rest-mass m_π decays into an electron of rest-mass m_e and a neutrino of zero rest mass,

$$\pi^- \to e^- + \nu \, .$$

Show that v, the speed of the electron in the rest frame of the π^-, is

$$v = c \left[\frac{1 - \left(m_e/m_\pi\right)^2}{1 + \left(m_e/m_\pi\right)^2} \right] \, .$$

18D Fluid Dynamics

Starting from Euler's equation for an inviscid, incompressible fluid in the absence of body forces,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p\,,$$

derive the equation for the vorticity $\boldsymbol{\omega} = \nabla_\wedge \mathbf{u}$.

[*You may assume that* $\nabla_\wedge(\mathbf{a}_\wedge\mathbf{b}) = \mathbf{a}\nabla.\mathbf{b} - \mathbf{b}\nabla.\mathbf{a} + (\mathbf{b}.\nabla)\mathbf{a} - (\mathbf{a}.\nabla)\mathbf{b}$.]

Show that, in a two-dimensional flow, vortex lines keep their strength and move with the fluid.

Show that a two-dimensional flow driven by a line vortex of circulation Γ at distance b from a rigid plane wall is the same as if the wall were replaced by another vortex of circulation $-\Gamma$ at the image point, distance b from the wall on the other side. Deduce that the first vortex will move at speed $\Gamma/4\pi b$ parallel to the wall.

A line vortex of circulation Γ moves in a quarter-plane, bounded by rigid plane walls at $x = 0,\ y > 0$ and $y = 0,\ x > 0$. Show that the vortex follows a trajectory whose equation in plane polar coordinates is $r \sin 2\theta = \text{constant}$.

19C Statistics

Consider the linear regression model

$$Y_i = \alpha + \beta x_i + \epsilon_i, \qquad 1 \leqslant i \leqslant n\,,$$

where $\epsilon_1, \ldots, \epsilon_n$ are independent, identically distributed $N(0, \sigma^2)$, $x_1, \ldots, x_n$ are known real numbers with $\sum_{i=1}^n x_i = 0$ and α, β and σ^2 are unknown.

(i) Find the least-squares estimates $\widehat{\alpha}$ and $\widehat{\beta}$ of α and β, respectively, and explain why in this case they are the same as the maximum-likelihood estimates.

(ii) Determine the maximum-likelihood estimate $\widehat{\sigma}^2$ of σ^2 and find a multiple of it which is an unbiased estimate of σ^2.

(iii) Determine the joint distribution of $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\sigma}^2$.

(iv) Explain carefully how you would test the hypothesis $H_0 : \alpha = \alpha_0$ against the alternative $H_1 : \alpha \neq \alpha_0$.

20C Optimization

Consider the linear programming problem

$$\begin{array}{rrl} \text{minimize} & 2x_1 - 3x_2 - 2x_3 & \\ \text{subject to} & -2x_1 + 2x_2 + 4x_3 & \leqslant 5 \\ & 4x_1 + 2x_2 - 5x_3 & \leqslant 8 \\ & 5x_1 - 4x_2 + \tfrac{1}{2}x_3 & \leqslant 5, \qquad x_i \geqslant 0, \quad i = 1, 2, 3. \end{array}$$

(i) After adding slack variables z_1, z_2 and z_3 and performing one iteration of the simplex algorithm, the following tableau is obtained.

	x_1	x_2	x_3	z_1	z_2	z_3	
x_2	-1	1	2	1/2	0	0	5/2
z_2	6	0	-9	-1	1	0	3
z_3	1	0	17/2	2	0	1	15
Payoff	-1	0	4	3/2	0	0	15/2

Complete the solution of the problem.

(ii) Now suppose that the problem is amended so that the objective function becomes

$$2x_1 - 3x_2 - 5x_3.$$

Find the solution of this new problem.

(iii) Formulate the dual of the problem in (ii) and identify the optimal solution to the dual.

END OF PAPER