UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday 5 June 2007 9 to 12

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

Suppose that $\{e_1, \ldots, e_3\}$ is a basis of the complex vector space $\mathbb{C}^3$ and that $A : \mathbb{C}^3 \to \mathbb{C}^3$ is the linear operator defined by $A(e_1) = e_2$, $A(e_2) = e_3$, and $A(e_3) = e_1$.

By considering the action of A on column vectors of the form $(1, \xi, \xi^2)^T$, where $\xi^3 = 1$, or otherwise, find the diagonalization of A and its characteristic polynomial.

2A Geometry

State the Gauss–Bonnet theorem for spherical triangles, and deduce from it that for each convex polyhedron with F faces, E edges, and V vertices, $F - E + V = 2$.

3F Complex Analysis or Complex Methods

For the function

$$f(z) = \frac{2z}{z^2 + 1} \,,$$

determine the Taylor series of f around the point $z_0 = 1$, and give the largest r for which this series converges in the disc $|z - 1| < r$.

4B Special Relativity

Write down the position four-vector. Suppose this represents the position of a particle with rest mass M and velocity $\mathbf{v}$. Show that the four momentum of the particle is

$$p_a = (M\gamma c,\ M\gamma\mathbf{v})\,,$$

where $\gamma = \left(1 - |\mathbf{v}|^2/c^2\right)^{-1/2}$.

For a particle of zero rest mass show that

$$p_a = (|\mathbf{p}|,\ \mathbf{p})\,,$$

where $\mathbf{p}$ is the three momentum.

5D Fluid Dynamics

A steady two-dimensional velocity field is given by

$$\mathbf{u}(x,y) = (\alpha x - \beta y\,,\, \beta x - \alpha y)\,, \quad \alpha > 0\,, \quad \beta > 0\,.$$

(i) Calculate the vorticity of the flow.

(ii) Verify that $\mathbf{u}$ is a possible flow field for an incompressible fluid, and calculate the stream function.

(iii) Show that the streamlines are bounded if and only if $\alpha < \beta$.

(iv) What are the streamlines in the case $\alpha = \beta$?

6F Numerical Analysis

Solve the least squares problem

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

using QR method with Householder transformation. (A solution using normal equations is *not* acceptable.)

7C Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed random variables from the $N\left(\mu, \sigma^2\right)$ distribution where μ and σ^2 are unknown. Use the generalized likelihood-ratio test to derive the form of a test of the hypothesis $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$.

Explain carefully how the test should be implemented.

8C Optimization

State and prove the max-flow min-cut theorem for network flows.

SECTION II

9G Linear Algebra

State and prove Sylvester's law of inertia for a real quadratic form.

[*You may assume that for each real symmetric matrix A there is an orthogonal matrix U, such that $U^{-1}AU$ is diagonal.*]

Suppose that V is a real vector space of even dimension $2m$, that Q is a non-singular quadratic form on V and that U is an m-dimensional subspace of V on which Q vanishes. What is the signature of Q?

10G Groups, Rings and Modules

(i) State a structure theorem for finitely generated abelian groups.

(ii) If K is a field and f a polynomial of degree n in one variable over K, what is the maximal number of zeroes of f? Justify your answer in terms of unique factorization in some polynomial ring, or otherwise.

(iii) Show that any finite subgroup of the multiplicative group of non-zero elements of a field is cyclic. Is this true if the subgroup is allowed to be infinite?

11H Analysis II

Define what it means for a function $f : \mathbb{R}^a \to \mathbb{R}^b$ to be differentiable at a point $p \in \mathbb{R}^a$ with derivative a linear map $Df|_p$.

State the Chain Rule for differentiable maps $f : \mathbb{R}^a \to \mathbb{R}^b$ and $g : \mathbb{R}^b \to \mathbb{R}^c$. Prove the Chain Rule.

Let $\|x\|$ denote the standard Euclidean norm of $x \in \mathbb{R}^a$. Find the partial derivatives $\frac{\partial f}{\partial x_i}$ of the function $f(x) = \|x\|$ where they exist.

12A Metric and Topological Spaces

Let X and Y be topological spaces. Define the product topology on $X \times Y$ and show that if X and Y are Hausdorff then so is $X \times Y$.

Show that the following statements are equivalent.

(i) X is a Hausdorff space.

(ii) The diagonal $\Delta = \{(x,x) : x \in X\}$ is a closed subset of $X \times X$, in the product topology.

(iii) For any topological space Y and any continuous maps $f, g : Y \to X$, the set $\{y \in Y : f(y) = g(y)\}$ is closed in Y.

13F Complex Analysis or Complex Methods

By integrating round the contour C_R, which is the boundary of the domain

$$D_R = \{z = re^{i\theta} : 0 < r < R, \quad 0 < \theta < \frac{\pi}{4}\},$$

evaluate each of the integrals

$$\int_0^\infty \sin x^2 dx, \quad \int_0^\infty \cos x^2 dx\,.$$

[*You may use the relations* $\int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2}$ *and* $\sin t \geq \frac{2}{\pi} t$ *for* $0 \leq t \leq \frac{\pi}{2}$ *.*]

14D Methods

Define the Fourier transform $\tilde{f}(k)$ of a function $f(x)$ that tends to zero as $|x| \to \infty$, and state the inversion theorem. State and prove the convolution theorem.

Calculate the Fourier transforms of

$$\begin{aligned} &(i) \quad f(x) = e^{-a|x|}, \\ \text{and} \quad &(ii) \quad g(x) = \begin{cases} 1\,, & |x| \leqslant b \\ 0\,, & |x| > b\,. \end{cases} \end{aligned}$$

Hence show that

$$\int_{-\infty}^{\infty} \frac{\sin(bk)\, e^{ikx}}{k\,(a^2 + k^2)}\, dk = \frac{\pi \sinh(ab)}{a^2}\, e^{-ax} \qquad \text{for} \quad x > b\,,$$

and evaluate this integral for all other (real) values of x.

15B Quantum Mechanics

The relative motion of a neutron and proton is described by the Schrödinger equation for a single particle of mass m under the influence of the central potential

$$V(r) = \begin{cases} -U & r < a \\ 0 & r > a, \end{cases}$$

where U and a are positive constants. Solve this equation for a spherically symmetric state of the deuteron, which is a bound state of a proton and neutron, giving the condition on U for this state to exist.

[*If* ψ *is spherically symmetric then* $\nabla^2\psi = \frac{1}{r}\frac{d^2}{dr^2}(r\psi)$.]

16E Electromagnetism

A steady magnetic field $\mathbf{B}(\mathbf{x})$ is generated by a current distribution $\mathbf{j}(\mathbf{x})$ that vanishes outside a bounded region V. Use the divergence theorem to show that

$$\int_V \mathbf{j}\, dV = 0 \quad \text{and} \quad \int_V x_i j_k\, dV = -\int_V x_k j_i\, dV.$$

Define the *magnetic vector potential* $\mathbf{A}(\mathbf{x})$. Use Maxwell's equations to obtain a differential equation for $\mathbf{A}(\mathbf{x})$ in terms of $\mathbf{j}(\mathbf{x})$.

It may be shown that for an unbounded domain the equation for $\mathbf{A}(\mathbf{x})$ has solution

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, dV'\,.$$

Deduce that in general the leading order approximation for $\mathbf{A}(\mathbf{x})$ as $|\mathbf{x}| \to \infty$ is

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3} \quad \text{where} \quad \mathbf{m} = \frac{1}{2}\int_V \mathbf{x}'\times\mathbf{j}(\mathbf{x}')\, dV'\,.$$

Find the corresponding far-field expression for $\mathbf{B}(\mathbf{x})$.

17D Fluid Dynamics

Write down the Euler equation for the steady motion of an inviscid, incompressible fluid in a constant gravitational field. From this equation, derive (a) Bernoulli's equation and (b) the integral form of the momentum equation for a fixed control volume V with surface S.

(i) A circular jet of water is projected vertically upwards with speed U_0 from a nozzle of cross-sectional area A_0 at height $z = 0$. Calculate how the speed U and cross-sectional area A of the jet vary with z, for $z \ll U_0^2/2g$.

(ii) A circular jet of speed U and cross-sectional area A impinges axisymmetrically on the vertex of a cone of semi-angle α, spreading out to form an almost parallel-sided sheet on the surface. Choose a suitable control volume and, neglecting gravity, show that the force exerted by the jet on the cone is

$$\rho A U^2 (1 - \cos\alpha)\,.$$

(iii) A cone of mass M is supported, axisymmetrically and vertex down, by the jet of part (i), with its vertex at height $z = h$, where $h \ll U_0^2/2g$. Assuming that the result of part (ii) still holds, show that h is given by

$$\rho A_0 U_0^2 \left(1 - \frac{2gh}{U_0^2}\right)^{\frac{1}{2}} (1 - \cos\alpha) \;=\; Mg\,.$$

18C Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed random variables with

$$\mathbb{P}(X_i = 1) = \theta = 1 - \mathbb{P}(X_i = 0) \ ,$$

where θ is an unknown parameter, $0 < \theta < 1$, and $n \geqslant 2$. It is desired to estimate the quantity $\phi = \theta(1-\theta) = n\mathbb{V}\mathrm{ar}\left((X_1 + \cdots + X_n)/n\right)$.

(i) Find the maximum-likelihood estimate, $\hat{\phi}$, of ϕ.

(ii) Show that $\hat{\phi}_1 = X_1(1 - X_2)$ is an unbiased estimate of ϕ and hence, or otherwise, obtain an unbiased estimate of ϕ which has smaller variance than $\hat{\phi}_1$ and which is a function of $\hat{\phi}$.

(iii) Now suppose that a Bayesian approach is adopted and that the prior distribution for θ, $\pi(\theta)$, is taken to be the uniform distribution on $(0,1)$. Compute the Bayes point estimate of ϕ when the loss function is $L(\phi, a) = (\phi - a)^2$.

[*You may use that fact that when r, s are non-negative integers,*

$$\int_0^1 x^r (1-x)^s dx = r!s!/(r+s+1)! \quad]$$

19C Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ on states $\{0, 1, \ldots, r\}$ with transition matrix (P_{ij}), where $P_{0,0} = 1 = P_{r,r}$, so that 0 and r are absorbing states. Let

$$A = (X_n = 0, \text{ for some } n \geqslant 0) \ ,$$

be the event that the chain is absorbed in 0. Assume that $h_i = \mathbb{P}(A \mid X_0 = i) > 0$ for $1 \leqslant i < r$.

Show carefully that, conditional on the event A, $(X_n)_{n \geqslant 0}$ is a Markov chain and determine its transition matrix.

Now consider the case where $P_{i,i+1} = \frac{1}{2} = P_{i,i-1}$, for $1 \leqslant i < r$. Suppose that $X_0 = i$, $1 \leqslant i < r$, and that the event A occurs; calculate the expected number of transitions until the chain is first in the state 0.

END OF PAPER