UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 4th June 2007 1.30 pm to 4.30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***C*** *and* ***E*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

(i) Use Euclid's algorithm to find all pairs of integers x and y such that

$$7x + 18y = 1\,.$$

(ii) Show that, if n is odd, then $n^3 - n$ is divisible by 24.

2E Numbers and Sets

For integers k and n with $0 \leqslant k \leqslant n$, define $\binom{n}{k}$. Arguing from your definition, show that

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

for all integers k and n with $1 \leqslant k \leqslant n-1$.

Use induction on k to prove that

$$\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$$

for all non-negative integers k and n.

3C Dynamics

A rocket, moving vertically upwards, ejects gas vertically downwards at speed u relative to the rocket. Derive, giving careful explanations, the equation of motion

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - gm\,,$$

where v and m are the speed and total mass of the rocket (including fuel) at time t.

If u is constant and the rocket starts from rest with total mass m_0, show that

$$m = m_0 e^{-(gt+v)/u}\,.$$

4C Dynamics

Sketch the graph of $y = 3x^2 - 2x^3$.

A particle of unit mass moves along the x axis in the potential $V(x) = 3x^2 - 2x^3$. Sketch the phase plane, and describe briefly the motion of the particle on the different trajectories.

SECTION II

5E Numbers and Sets

State and prove the Inclusion–Exclusion principle.

The keypad on a cash dispenser is broken. To withdraw money, a customer is required to key in a 4-digit number. However, the key numbered 0 will only function if either the immediately preceding two keypresses were both 1, or the very first key pressed was 2. Explaining your reasoning clearly, use the Inclusion–Exclusion Principle to find the number of 4-digit codes which can be entered.

6E Numbers and Sets

Stating carefully any results about countability you use, show that for any $d \geq 1$ the set $\mathbb{Z}[X_1, \ldots, X_d]$ of polynomials with integer coefficients in d variables is countable. By taking $d = 1$, deduce that there exist uncountably many transcendental numbers.

Show that there exists a sequence $x_1, x_2, \ldots$ of real numbers with the property that $f(x_1, \ldots, x_d) \neq 0$ for every $d \geq 1$ and for every non-zero polynomial $f \in \mathbb{Z}[X_1, \ldots, X_d]$.

[*You may assume without proof that* $\mathbb{R}$ *is uncountable.*]

7E Numbers and Sets

Let x_n $(n = 1, 2, \ldots)$ be real numbers.

What does it mean to say that the sequence $(x_n)_{n=1}^{\infty}$ converges?

What does it mean to say that the series $\sum_{n=1}^{\infty} x_n$ converges?

Show that if $\sum_{n=1}^{\infty} x_n$ is convergent, then $x_n \to 0$. Show that the converse can be false.

Sequences of positive real numbers x_n, y_n $(n \geq 1)$ are given, such that the inequality

$$y_{n+1} \leq y_n - \frac{1}{2}\min(x_n, y_n)$$

holds for all $n \geq 1$. Show that, if $\sum_{n=1}^{\infty} x_n$ diverges, then $y_n \to 0$.

8E Numbers and Sets

(i) Let p be a prime number, and let x and y be integers such that p divides xy. Show that at least one of x and y is divisible by p. Explain how this enables one to prove the Fundamental Theorem of Arithmetic.

[*Standard properties of highest common factors may be assumed without proof.*]

(ii) State and prove the Fermat-Euler Theorem.

Let $1/359$ have decimal expansion $0 \cdot a_1 a_2 \dots$ with $a_n \in \{0, 1, \dots, 9\}$. Use the fact that $60^2 \equiv 10 \pmod{359}$ to show that, for every n, $a_n = a_{n+179}$.

9C Dynamics

A small ring of mass m is threaded on a smooth rigid wire in the shape of a parabola given by $x^2 = 4az$, where x measures horizontal distance and z measures distance vertically upwards. The ring is held at height $z = h$, then released.

(i) Show by dimensional analysis that the period of oscillations, T, can be written in the form

$$T = (a/g)^{1/2} G(h/a)$$

for some function G.

(ii) Show that G is given by

$$G(\beta) = 2\sqrt{2} \int_{-1}^{1} \left(\frac{1 + \beta u^2}{1 - u^2} \right)^{\frac{1}{2}} du \,,$$

and find, to first order in h/a, the period of small oscillations.

10C Dynamics

A particle of mass m experiences, at the point with position vector $\mathbf{r}$, a force $\mathbf{F}$ given by

$$\mathbf{F} = -k\mathbf{r} - e\dot{\mathbf{r}} \times \mathbf{B} \,,$$

where k and e are positive constants and $\mathbf{B}$ is a constant, uniform, vector field.

(i) Show that $m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + k\mathbf{r} \cdot \mathbf{r}$ is constant. Give a physical interpretation of each term and a physical explanation of the fact that $\mathbf{B}$ does not arise in this expression.

(ii) Show that $m(\dot{\mathbf{r}} \times \mathbf{r}) \cdot \mathbf{B} + \frac{1}{2} e(\mathbf{r} \times \mathbf{B}) \cdot (\mathbf{r} \times \mathbf{B})$ is constant.

(iii) Given that the particle was initially at rest at $\mathbf{r}_0$, derive an expression for $\mathbf{r} \cdot \mathbf{B}$ at time t.

11C Dynamics

A particle moves in the gravitational field of the Sun. The angular momentum per unit mass of the particle is h and the mass of the Sun is M. Assuming that the particle moves in a plane, write down the equations of motion in polar coordinates, and derive the equation

$$\frac{d^2u}{d\theta^2} + u = k\,,$$

where $u = 1/r$ and $k = GM/h^2$.

Write down the equation of the orbit (u as a function of θ), given that the particle moves with the escape velocity and is at the perihelion of its orbit, a distance r_0 from the Sun, when $\theta = 0$. Show that

$$\sec^4(\theta/2)\frac{d\theta}{dt} = \frac{h}{r_0^2}$$

and hence that the particle reaches a distance $2r_0$ from the Sun at time $8r_0^2/(3h)$.

12C Dynamics

The ith particle of a system of N particles has mass m_i and, at time t, position vector $\mathbf{r}_i$ with respect to an origin O. It experiences an external force $\mathbf{F}_i^e$, and also an internal force $\mathbf{F}_{ij}$ due to the jth particle (for each $j = 1, ..., N$, $j \neq i$), where $\mathbf{F}_{ij}$ is parallel to $\mathbf{r}_i - \mathbf{r}_j$ and Newton's third law applies.

(i) Show that the position of the centre of mass, $\mathbf{X}$, satisfies

$$M \frac{d^2\mathbf{X}}{dt^2} = \mathbf{F}^e \,,$$

where M is the total mass of the system and $\mathbf{F}^e$ is the sum of the external forces.

(ii) Show that the total angular momentum of the system about the origin, $\mathbf{L}$, satisfies

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} \,,$$

where $\mathbf{N}$ is the total moment about the origin of the external forces.

(iii) Show that $\mathbf{L}$ can be expressed in the form

$$\mathbf{L} = M\mathbf{X} \times \mathbf{V} + \sum_i m_i \mathbf{r}_i' \times \mathbf{v}_i' \,,$$

where $\mathbf{V}$ is the velocity of the centre of mass, $\mathbf{r}_i'$ is the position vector of the ith particle relative to the centre of mass, and $\mathbf{v}_i'$ is the velocity of the ith particle relative to the centre of mass.

(iv) In the case $N = 2$ when the internal forces are derived from a potential $U(|\mathbf{r}|)$, where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and there are no external forces, show that

$$\frac{dT}{dt} + \frac{dU}{dt} = 0 \,,$$

where T is the total kinetic energy of the system.

END OF PAPER