UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 1st June 2007 1.30 pm to 4.30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***B*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Differential Equations

Find the solution $y(x)$ of the equation

$$y'' - 6y' + 9y = \cos(2x)\, \mathrm{e}^{3x}$$

that satisfies $y(0) = 0$ and $y'(0) = 1$.

2B Differential Equations

Investigate the stability of:

(i) the equilibrium points of the equation

$$\frac{dy}{dt} = (y^2 - 4)\tan^{-1}(y)\,;$$

(ii) the constant solutions ($u_{n+1} = u_n$) of the discrete equation

$$u_{n+1} = \frac{1}{2}u_n^2(1 + u_n)\,.$$

3F Probability

Let X and Y be independent random variables, each uniformly distributed on $[0,1]$. Let $U = \min(X,Y)$ and $V = \max(X,Y)$. Show that $\mathbb{E}U = \frac{1}{3}$, and hence find the covariance of U and V.

4F Probability

Let X be a normally distributed random variable with mean 0 and variance 1. Define, and determine, the moment generating function of X. Compute $\mathbb{E}X^r$ for $r = 0, 1, 2, 3, 4$.

Let Y be a normally distributed random variable with mean μ and variance σ^2. Determine the moment generating function of Y.

SECTION II

5B Differential Equations

(i) The function $y(z)$ satisfies the equation

$$y'' + p(z)y' + q(z)y = 0\,.$$

Give the definitions of the terms *ordinary point, singular point,* and *regular singular point* for this equation.

(ii) For the equation

$$4zy'' + 2y' + y = 0\,,$$

classify the point $z = 0$ according to the definitions you gave in (i), and find the series solutions about $z = 0$. Identify these solutions in closed form.

6B Differential Equations

Find the most general solution of the equation

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 1$$

by making the change of variables

$$\xi = x + 2y, \qquad \eta = x + 3y\,.$$

Find the solution that satisfies $u = 0$ and $\partial u/\partial y = x$ when $y = 0$.

7B Differential Equations

(i) Find, in the form of an integral, the solution of the equation

$$\alpha\frac{dy}{dt} + y = f(t)$$

that satisfies $y \to 0$ as $t \to -\infty$. Here $f(t)$ is a general function and α is a positive constant.

Hence find the solution in each of the cases:

(a) $f(t) = \delta(t)$;

(b) $f(t) = H(t)$, where $H(t)$ is the Heaviside step function.

(ii) Find and sketch the solution of the equation

$$\frac{dy}{dt} + y = H(t) - H(t-1)\,,$$

given that $y(0) = 0$ and $y(t)$ is continuous.

8B Differential Equations

(i) Find the general solution of the difference equation

$$u_{k+1} + 5u_k + 6u_{k-1} = 12k + 1\,.$$

(ii) Find the solution of the equation

$$y_{k+1} + 5y_k + 6y_{k-1} = 2^k$$

that satisfies $y_0 = y_1 = 1$. Hence show that, for any positive integer n, the quantity $2^n - 26(-3)^n$ is divisible by 10.

9F Probability

Let N be a non-negative integer-valued random variable with

$$P\{N = r\} = p_r, \quad r = 0, 1, 2, \ldots$$

Define $\mathbb{E}N$, and show that

$$\mathbb{E}N = \sum_{n=1}^{\infty} P\{N \geqslant n\}\,.$$

Let X_1, X_2, $\ldots$ be a sequence of independent and identically distributed continuous random variables. Let the random variable N mark the point at which the sequence stops decreasing: that is, $N \geqslant 2$ is such that

$$X_1 \geqslant X_2 \geqslant \ldots \geqslant X_{N-1} < X_N\,,$$

where, if there is no such finite value of N, we set $N = \infty$. Compute $P\{N = r\}$, and show that $P\{N = \infty\} = 0$. Determine $\mathbb{E}N$.

10F Probability

Let X and Y be independent non-negative random variables, with densities f and g respectively. Find the joint density of $U = X$ and $V = X + aY$, where a is a positive constant.

Let X and Y be independent and exponentially distributed random variables, each with density

$$f(x) = \lambda e^{-\lambda x}, \quad x \geqslant 0\,.$$

Find the density of $X + \frac{1}{2}Y$. Is it the same as the density of the random variable $\max(X, Y)$?

11F Probability

Let A_1, A_2, ..., A_n ($n \geqslant 2$) be events in a sample space. For each of the following statements, either prove the statement or provide a counterexample.

(i)

$$P\left(\bigcap_{k=2}^{n} A_k \,\middle|\, A_1\right) = \prod_{k=2}^{n} P\left(A_k \,\middle|\, \bigcap_{r=1}^{k-1} A_r\right), \quad \text{provided} \quad P\left(\bigcap_{k=1}^{n-1} A_k\right) > 0 .$$

(ii)

$$\text{If} \quad \sum_{k=1}^{n} P(A_k) > n-1 \quad \text{then} \quad P\left(\bigcap_{k=1}^{n} A_k\right) > 0 .$$

(iii)

$$\text{If} \quad \sum_{i<j} P\left(A_i \bigcap A_j\right) > \binom{n}{2} - 1 \quad \text{then} \quad P\left(\bigcap_{k=1}^{n} A_k\right) > 0 .$$

(iv) If B is an event and if, for each k, $\{B, A_k\}$ is a pair of independent events, then $\{B, \cup_{k=1}^{n} A_k\}$ is also a pair of independent events.

12F Probability

Let A, B and C be three random points on a sphere with centre O. The positions of A, B and C are independent, and each is uniformly distributed over the surface of the sphere. Calculate the probability density function of the angle $\angle AOB$ formed by the lines OA and OB.

Calculate the probability that all three of the angles $\angle AOB$, $\angle AOC$ and $\angle BOC$ are acute. [**Hint:** *Condition on the value of the angle* $\angle AOB$.]

END OF PAPER