MATHEMATICAL TRIPOS Part IA

Thursday 31st May 2007 9 am to 12 noon

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Algebra and Geometry

(i) The spherical polar unit basis vectors $\mathbf{e}_r, \mathbf{e}_\phi$ and $\mathbf{e}_\theta$ in $\mathbb{R}^3$ are given in terms of the Cartesian unit basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ by

$$\begin{aligned}\mathbf{e}_r &= \mathbf{i}\cos\phi\sin\theta + \mathbf{j}\sin\phi\sin\theta + \mathbf{k}\cos\theta\,,\\ \mathbf{e}_\theta &= \mathbf{i}\cos\phi\cos\theta + \mathbf{j}\sin\phi\cos\theta - \mathbf{k}\sin\theta\,,\\ \mathbf{e}_\phi &= -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi\,.\end{aligned}$$

Express $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ in terms of $\mathbf{e}_r, \mathbf{e}_\phi$ and $\mathbf{e}_\theta$.

(ii) Use suffix notation to prove the following identity for the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in $\mathbb{R}^3$:

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\,\mathbf{A}\,.$$

2B Algebra and Geometry

For the equations

$$\begin{aligned}px + y + z &= 1,\\ x + 2y + 4z &= t,\\ x + 4y + 10z &= t^2,\end{aligned}$$

find the values of p and t for which

(i) there is a unique solution;

(ii) there are infinitely many solutions;

(iii) there is no solution.

3F Analysis

Prove that, for positive real numbers a and b,

$$2\sqrt{ab} \leqslant a + b\,.$$

For positive real numbers $a_1, a_2, \ldots$, prove that the convergence of

$$\sum_{n=1}^{\infty} a_n$$

implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}\,.$$

4D Analysis

Let $\sum_{n=0}^{\infty} a_n z^n$ be a complex power series. Show that there exists $R \in [0, \infty]$ such that $\sum_{n=0}^{\infty} a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$.

Find the value of R for the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}\,.$$

SECTION II

5B Algebra and Geometry

(i) Describe geometrically the following surfaces in three-dimensional space:

(a) $\mathbf{r} \cdot \mathbf{u} = \alpha|\mathbf{r}|$, where $0 < |\alpha| < 1$;

(b) $|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}| = \beta$, where $\beta > 0$.

Here α and β are fixed scalars and $\mathbf{u}$ is a fixed unit vector. You should identify the meaning of α, β and $\mathbf{u}$ for these surfaces.

(ii) The plane $\mathbf{n} \cdot \mathbf{r} = p$, where $\mathbf{n}$ is a fixed unit vector, and the sphere with centre $\mathbf{c}$ and radius a intersect in a circle with centre $\mathbf{b}$ and radius ρ.

(a) Show that $\mathbf{b} - \mathbf{c} = \lambda\mathbf{n}$, where you should give λ in terms of a and ρ.

(b) Find ρ in terms of $\mathbf{c}, \mathbf{n}, a$ and p.

6C Algebra and Geometry

Let $\mathcal{M} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$\mathbf{x} \mapsto \mathbf{x}' = a\mathbf{x} + b(\mathbf{n} \times \mathbf{x}) \,,$$

where a and b are positive scalar constants, and $\mathbf{n}$ is a unit vector.

(i) By considering the effect of $\mathcal{M}$ on $\mathbf{n}$ and on a vector orthogonal to $\mathbf{n}$, describe geometrically the action of $\mathcal{M}$.

(ii) Express the map $\mathcal{M}$ as a matrix M using suffix notation. Find a, b and $\mathbf{n}$ in the case

$$\mathrm{M} = \begin{pmatrix} 2 & -2 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & 2 \end{pmatrix} .$$

(iii) Find, in the general case, the inverse map (i.e. express $\mathbf{x}$ in terms of $\mathbf{x}'$ in vector form).

7C Algebra and Geometry

Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in a real vector space with scalar product denoted by $\mathbf{x}\cdot\mathbf{y}$. Prove that $(\mathbf{x}\cdot\mathbf{y})^{\mathbf{2}} \leq (\mathbf{x}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y})$, and prove also that $(\mathbf{x}\cdot\mathbf{y})^{\mathbf{2}} = (\mathbf{x}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y})$ if and only if $\mathbf{x} = \lambda\mathbf{y}$ for some scalar λ.

(i) By considering suitable vectors in $\mathbb{R}^3$, or otherwise, prove that the inequality $x^2 + y^2 + z^2 \geq yz + zx + xy$ holds for any real numbers x, y and z.

(ii) By considering suitable vectors in $\mathbb{R}^4$, or otherwise, show that only one choice of real numbers x, y, z satisfies $3(x^2 + y^2 + z^2 + 4) - 2(yz + zx + xy) - 4(x + y + z) = 0$, and find these numbers.

8A Algebra and Geometry

(i) Show that any line in the complex plane $\mathbb{C}$ can be represented in the form

$$\bar{c}z + c\bar{z} + r = 0\,,$$

where $c \in \mathbb{C}$ and $r \in \mathbb{R}$.

(ii) If z and u are two complex numbers for which

$$\left|\frac{z+u}{z+\bar{u}}\right| = 1\,,$$

show that either z or u is real.

(iii) Show that any Möbius transformation

$$w = \frac{az+b}{cz+d} \quad (bc - ad \neq 0)$$

that maps the real axis $z = \bar{z}$ into the unit circle $|w| = 1$ can be expressed in the form

$$w = \lambda\frac{z+k}{z+\bar{k}}\,,$$

where $\lambda, k \in \mathbb{C}$ and $|\lambda| = 1$.

9F Analysis

Let $a_1 = \sqrt{2}$, and consider the sequence of positive real numbers defined by

$$a_{n+1} = \sqrt{2 + \sqrt{a_n}}\,, \qquad n = 1, 2, 3, \ldots\,.$$

Show that $a_n \leqslant 2$ for all n. Prove that the sequence $a_1, a_2, \ldots$ converges to a limit.

Suppose instead that $a_1 = 4$. Prove that again the sequence $a_1, a_2, \ldots$ converges to a limit.

Prove that the limits obtained in the two cases are equal.

10E Analysis

State and prove the Mean Value Theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that, for every $x \in \mathbb{R}$, $f''(x)$ exists and is non-negative.

(i) Show that if $x \leq y$ then $f'(x) \leq f'(y)$.

(ii) Let $\lambda \in (0,1)$ and $a < b$. Show that there exist x and y such that

$$f\big(\lambda a + (1-\lambda)b\big) = f(a) + (1-\lambda)(b-a)f'(x) = f(b) - \lambda(b-a)f'(y)$$

and that

$$f\big(\lambda a + (1-\lambda)b\big) \leq \lambda f(a) + (1-\lambda)f(b)\,.$$

11E Analysis

Let $a < b$ be real numbers, and let $f : [a,b] \to \mathbb{R}$ be continuous. Show that f is bounded on $[a,b]$, and that there exist $c, d \in [a,b]$ such that for all $x \in [a,b]$, $f(c) \leq f(x) \leq f(d)$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{x \to +\infty} g(x) = \lim_{x \to -\infty} g(x) = 0\,.$$

Show that g is bounded. Show also that, if a and c are real numbers with $0 < c \leq g(a)$, then there exists $x \in \mathbb{R}$ with $g(x) = c$.

12D Analysis

Explain carefully what it means to say that a bounded function $f : [0,1] \to \mathbb{R}$ is *Riemann integrable.*

Prove that every continuous function $f : [0,1] \to \mathbb{R}$ is Riemann integrable.

For each of the following functions from $[0,1]$ to $\mathbb{R}$, determine with proof whether or not it is Riemann integrable:

(i) the function $f(x) = x \sin \dfrac{1}{x}$ for $x \neq 0$, with $f(0) = 0$;

(ii) the function $g(x) = \sin \dfrac{1}{x}$ for $x \neq 0$, with $g(0) = 0$.

END OF PAPER