UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday 8 June 2006 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,*...*,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Let $N = p_1 p_2 \ldots p_r$ be a product of distinct primes, and let $\lambda(N)$ be the least common multiple of $p_1 - 1, p_2 - 1, \ldots, p_r - 1$. Prove that

$$a^{\lambda(N)} \equiv 1 \bmod N \quad \text{when} \quad (a, N) = 1.$$

Now take $N = 7 \times 13 \times 19$, and prove that

$$a^{N-1} \equiv 1 \bmod N \quad \text{when} \quad (a, N) = 1.$$

2G Topics in Analysis

Let a_0, a_1, a_2, $\ldots$ be positive integers and, for each n, let

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}} \;,$$

with $(p_n, q_n) = 1$.

Obtain an expression for the matrix $\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$ and use it to show that $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n+1}$.

3F Geometry and Groups

Let G be a discrete subgroup of the Möbius group. Define the limit set of G in S^2. If G contains two loxodromic elements whose fixed point sets in S^2 are different, show that the limit set of G contains no isolated points.

4G Coding and Cryptography

What does it mean to say that a binary code C has length n, size m and minimum distance d? Let $A(n,d)$ be the largest value of m for which there exists an $[n,m,d]$-code. Prove that

$$\frac{2^n}{V(n,d-1)} \leqslant A(n,d) \leqslant \frac{2^n}{V(n,\lfloor\frac{1}{2}(d-1)\rfloor)}$$

where $V(n,r) = \displaystyle\sum_{j=0}^{r} \binom{n}{j}$.

5I Statistical Modelling

Consider a generalized linear model for independent observations $Y_1,\ldots,Y_n$, with $\mathbb{E}(Y_i) = \mu_i$ for $i = 1,\ldots,n$. What is a *linear predictor*? What is meant by the *link function*? If Y_i has model function (or density) of the form

$$f(y_i;\mu_i,\sigma^2) = \exp\left[\frac{1}{\sigma^2}\{\theta(\mu_i)y_i - K(\theta(\mu_i))\}\right] a(\sigma^2,y_i),$$

for $y_i \in \mathcal{Y} \subseteq \mathbb{R}$, $\mu_i \in \mathcal{M} \subseteq \mathbb{R}$, $\sigma^2 \in \Phi \subseteq (0,\infty)$, where $a(\sigma^2,y_i)$ is a known positive function, define the *canonical link function*.

Now suppose that $Y_1,\ldots,Y_n$ are independent with $Y_i \sim \text{Bin}(1,\mu_i)$ for $i = 1,\ldots,n$. Derive the canonical link function.

6B Mathematical Biology

The SIR epidemic model for an infectious disease divides the population N into three categories of *susceptible* $S(t)$, *infected* $I(t)$ and *recovered* (non-infectious) $R(t)$. It is supposed that the disease is non-lethal, so that the population does not change in time.

Explain the reasons for the terms in the following model equations:

$$\frac{dS}{dt} = pR - rIS, \qquad \frac{dI}{dt} = rIS - aI, \qquad \frac{dR}{dt} = aI - pR.$$

At time $t = 0$, $S \approx N$ while $I, R \ll 1$.

(a) Show that if $rN < a$ no epidemic occurs.

(b) Now suppose that $p > 0$ and there is an epidemic. Show that the system has a non-trivial fixed point, and that it is stable to small disturbances. Show also that for both small and large p both the trace and the determinant of the Jacobian matrix are $O(p)$, and deduce that the matrix has complex eigenvalues for sufficiently small p, and real eigenvalues for sufficiently large p.

7E Dynamical Systems

State the normal-form equations for (a) a saddle-node bifurcation, (b) a transcritical bifurcation, and (c) a pitchfork bifurcation, for a dynamical system $\dot{x} = f(x, \mu)$.

Consider the system

$$\begin{aligned}\dot{x} &= \mu + y - x^2 + 2xy + 3y^2 \\ \dot{y} &= -y + 2x^2 + 3xy\ .\end{aligned}$$

Compute the extended centre manifold near $x = y = \mu = 0$, and the evolution equation on the centre manifold, both correct to second order in x and μ. Deduce the type of bifurcation and show that the equation can be put in normal form, to the same order, by a change of variables of the form $T = \alpha t$, $X = x - \beta\mu$, $\tilde{\mu} = \gamma(\mu)$ for suitably chosen α, β and $\gamma(\mu)$.

8E Further Complex Methods

Show that, for $b \neq 0$,

$$\mathcal{P}\int_0^\infty \frac{\cos u}{u^2 - b^2}\, du = -\frac{\pi}{2b}\sin b$$

where $\mathcal{P}$ denotes the Cauchy principal value.

9C Classical Dynamics

A pendulum of length ℓ oscillates in the xy plane, making an angle $\theta(t)$ with the vertical y axis. The pivot is attached to a moving lift that descends with constant acceleration a, so that the position of the bob is

$$x = \ell \sin\theta\ , \qquad y = \tfrac{1}{2}at^2 + \ell\cos\theta\ .$$

Given that the Lagrangian for an unconstrained particle is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy\ ,$$

determine the Lagrangian for the pendulum in terms of the generalized coordinate θ. Derive the equation of motion in terms of θ. What is the motion when $a = g$?

Find the equilibrium configurations for arbitrary a. Determine which configuration is stable when

(i) $a < g$

and when

(ii) $a > g$.

10D Cosmology

(a) Consider a spherically symmetric star with outer radius R, density $\rho(r)$ and pressure $P(r)$. By balancing the gravitational force on a shell at radius r against the force due to the pressure gradient, derive the pressure support equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,,$$

where $m(r) = \int_0^r \rho(r')\,4\pi r'^2\,dr'$. Show that this implies

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho\,.$$

Suggest appropriate boundary conditions at $r=0$ and $r=R$, together with a brief justification.

(b) Describe qualitatively the endpoint of stellar evolution for our sun when all its nuclear fuel is spent. Your discussion should briefly cover electron degeneracy pressure and the relevance of stability against inverse beta-decay.

[*Note that $m_n - m_p \approx 2.6\,m_e$, where m_n, m_p, m_e are the masses of the neutron, proton and electron respectively.*]

SECTION II

11H Number Theory

State the prime number theorem, and Dirichlet's theorem on primes in arithmetic progression.

If p is an odd prime number, prove that -1 is a quadratic residue modulo p if and only if $p \equiv 1 \mod 4$.

Let $p_1, \ldots, p_m$ be distinct prime numbers, and define

$$N_1 = 4p_1 \ldots p_m - 1, \quad N_2 = 4\left(p_1 \ldots p_m\right)^2 + 1.$$

Prove that N_1 has at least one prime factor which is congruent to $3 \bmod 4$, and that every prime factor of N_2 must be congruent to $1 \bmod 4$.

Deduce that there are infinitely many primes which are congruent to 1 mod 4, and infinitely many primes which are congruent to 3 mod 4.

12G Coding and Cryptography

Describe the RSA system with public key (N, e) and private key (N, d). Briefly discuss the possible advantages or disadvantages of taking (i) $e = 2^{16}+1$ or (ii) $d = 2^{16}+1$.

Explain how to factor N when both the private key and public key are known.

Describe the bit commitment problem, and briefly indicate how RSA can be used to solve it.

13B Mathematical Biology

A chemical system with concentrations $u(x,t), v(x,t)$ obeys the coupled reaction-diffusion equations

$$\frac{du}{dt} = ru + u^2 - uv + \kappa_1 \frac{d^2u}{dx^2} \ ,$$
$$\frac{dv}{dt} = s(u^2 - v) + \kappa_2 \frac{d^2v}{dx^2} \ ,$$

where r, s, κ_1, κ_2 are constants with s, κ_1, κ_2 positive.

(a) Find conditions on r, s such that there is a steady homogeneous solution $u = u_0$, $v = u_0^2$ which is stable to spatially homogeneous perturbations.

(b) Investigate the stability of this homogeneous solution to disturbances proportional to $\exp(ikx)$. Assuming that a solution satisfying the conditions of part (a) exists, find the region of parameter space in which the solution is stable to space-dependent disturbances, and show in particular that one boundary of this region for fixed s is given by

$$d \equiv \sqrt{\frac{\kappa_2}{\kappa_1}} = \sqrt{2s} + \frac{1}{u_0}\sqrt{s(2u_0^2 - u_0)} \ .$$

Sketch the various regions of existence and stability of steady, spatially homogeneous solutions in the (d, u_0) plane for the case $s = 2$.

(c) Show that the critical wavenumber $k = k_c$ for the onset of the instability satisfies the relation

$$k_c^2 = \frac{1}{\sqrt{\kappa_1 \kappa_2}} \left[\frac{s(d - \sqrt{2s})}{d(2\sqrt{2s} - d)} \right] .$$

Explain carefully what happens when $d < \sqrt{2s}$ and when $d > 2\sqrt{2s}$.

14E Further Complex Methods

It is given that the hypergeometric function $F(a,b;c;z)$ is the solution of the hypergeometric equation determined by the Papperitz symbol

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix} \qquad (*)$$

that is analytic at $z=0$ and satisfies $F(a,b;c;0)=1$, and that for $\mathrm{Re}(c-a-b)>0$

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ .$$

[*You may assume that a,b,c are such that $F(a,b;c;1)$ exists.*]

(a) Show, by manipulating Papperitz symbols, that

$$F(a,b;c;z)=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right) \qquad (|\arg(1-z)|<\pi).$$

(b) Let $w_1(z)=(-z)^{-a}F\left(a,1+a-c;1+a-b;\frac{1}{z}\right)$, where $|\arg(-z)|<\pi$. Show that $w_1(z)$ satisfies the hypergeometric equation determined by $(*)$.

(c) By considering the limit $z\to\infty$ in parts (a) and (b) above, deduce that, for $|\arg(-z)|<\pi$,

$$F(a,b;c;z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}w_1(z)+(\text{a similar term with } a \text{ and } b \text{ interchanged}).$$

15C Classical Dynamics

A particle of mass m is constrained to move on the surface of a sphere of radius ℓ. The Lagrangian is given in spherical polar coordinates by

$$L=\tfrac{1}{2}m\ell^2(\dot\theta^2+\dot\phi^2\sin^2\theta)+mg\ell\cos\theta\ ,$$

where gravity g is constant. Find the two constants of the motion.

The particle is projected horizontally with velocity v from a point whose depth below the centre is $\ell\cos\theta=D$. Find v such that the particle trajectory

(i) just grazes the horizontal equatorial plane $\theta=\pi/2$;

(ii) remains at depth D for all time t.

16H Logic and Set Theory

Explain what is meant by a *structure* for a first-order signature Σ, and describe how first-order formulae over Σ are interpreted in a given structure. Show that if B is a substructure of A, and ϕ is a quantifier-free formula (with n free variables), then

$$[\![\phi]\!]_B = [\![\phi]\!]_A \cap B^n .$$

A first-order theory is said to be *inductive* if its axioms all have the form

$$(\forall x_1, \ldots, x_n)(\exists y_1, \ldots, y_m)\phi$$

where ϕ is quantifier-free (and either of the strings $x_1, \ldots, x_n$ or $y_1, \ldots, y_m$ may be empty). If T is an inductive theory, and A is a structure for the appropriate signature, show that the poset of those substructures of A which are T-models is chain-complete.

Which of the following can be expressed as inductive theories over the signature with one binary predicate symbol $\leqslant$? Justify your answers.

(a) The theory of totally ordered sets without greatest or least elements.

(b) The theory of totally ordered sets with greatest and least elements.

17F Graph Theory

Let $R(s)$ be the least integer n such that every colouring of the edges of K_n with two colours contains a monochromatic K_s. Prove that $R(s)$ exists.

Prove that a connected graph of maximum degree $d \geqslant 2$ and order d^k contains two vertices distance at least k apart.

Let $C(s)$ be the least integer n such that every connected graph of order n contains, as an *induced* subgraph, either a complete graph K_s, a star $K_{1,s}$ or a path P_s of length s. Show that $C(s) \leqslant R(s)^s$.

18H Galois Theory

Let K be a field and m a positive integer, not divisible by the characteristic of K. Let L be the splitting field of the polynomial $X^m - 1$ over K. Show that $\mathrm{Gal}(L/K)$ is isomorphic to a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$.

Now assume that K is a finite field with q elements. Show that $[L : K]$ is equal to the order of the residue class of q in the group $(\mathbb{Z}/m\mathbb{Z})^*$. Hence or otherwise show that the splitting field of $X^{11} - 1$ over $\mathbb{F}_4$ has degree 5.

19F Representation Theory

(a) Let $G = \mathrm{SU}_2$, and let V_n be the space of homogeneous polynomials of degree n in the variables x and y. Thus $\dim V_n = n + 1$. Define the action of G on V_n and show that V_n is an irreducible representation of G.

(b) Decompose $V_3 \otimes V_3$ into irreducible representations. Decompose $\wedge^2 V_3$ and $\mathrm{S}^2 V_3$ into irreducible representations.

(c) Given any representation V of a group G, define the dual representation V^*. Show that V_n^* is isomorphic to V_n as a representation of SU_2.

[*You may use any results from the lectures provided that you state them clearly.*]

20H Algebraic Topology

Let X be the union of two circles identified at a point: the "figure eight". Classify all the connected double covering spaces of X. If we view these double coverings just as topological spaces, determine which of them are homeomorphic to each other and which are not.

21G Linear Analysis

Let X be a complex Banach space. We say a sequence $x^i \in X$ converges to $x \in X$ *weakly* if $\phi(x^i) \to \phi(x)$ for all $\phi \in X^*$. Let $T : X \to Y$ be bounded and linear. Show that if x^i converges to x weakly, then Tx^i converges to Tx weakly.

Now let $X = \ell_2$. Show that for a sequence $x^i \in X$, $i = 1, 2, \ldots$, with $||x^i|| \leqslant 1$, there exists a subsequence x^{i_k} such that x^{i_k} converges weakly to some $x \in X$ with $||x|| \leqslant 1$.

Now let $Y = \ell_1$, and show that $y^i \in Y$ converges to $y \in Y$ weakly if and only if $y^i \to y$ in the usual sense.

Define what it means for a linear operator $T : X \to Y$ to be *compact*, and deduce from the above that any bounded linear $T : \ell_2 \to \ell_1$ is compact.

22F Riemann Surfaces

Define the *branching order* $v_f(p)$ at a point p and the *degree* of a non-constant holomorphic map f between compact Riemann surfaces. State the Riemann–Hurwitz formula.

Let $W_m \subset \mathbb{C}^2$ be an affine curve defined by the equation $s^m = t^m + 1$, where $m \geqslant 2$ is an integer. Show that the projective curve $\overline{W}_m \subset \mathbb{P}^2$ corresponding to W_m is non-singular and identify the points of $\overline{W}_m \setminus W_m$. Let F be a continuous map from $\overline{W}_m$ to the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$, such that the restriction of F to W_m is given by $F(s,t) = s$. Show that F is holomorphic on $\overline{W}_m$. Find the degree and the ramification points of F on $\overline{W}_m$ and their branching orders. Determine the genus of $\overline{W}_m$.

[*Basic properties of the complex structure on an algebraic curve may be used without proof if accurately stated.*]

23H Differential Geometry

Let $S \subset \mathbb{R}^3$ be a connected oriented surface.

(a) Define the Gauss map $N : S \to S^2$ of S. Given $p \in S$, show that the derivative of N,

$$dN_p : T_pS \to T_{N(p)}S^2 = T_pS$$

is self-adjoint.

(b) Show that if N is a diffeomorphism, then the Gaussian curvature is positive everywhere. Is the converse true?

24J Probability and Measure

Let X be a real-valued random variable. Define the characteristic function ϕ_X. Show that $\phi_X(u) \in \mathbb{R}$ for all $u \in \mathbb{R}$ if and only if X and $-X$ have the same distribution.

For parts (a) and (b) below, let X and Y be independent and identically distributed random variables.

(a) Show that $X = Y$ almost surely implies that X is almost surely constant.

(b) Suppose that there exists $\varepsilon > 0$ such that $|\phi_X(u)| = 1$ for all $|u| < \varepsilon$. Calculate ϕ_{X-Y} to show that $\mathbb{E}\big(1 - \cos(u(X-Y))\big) = 0$ for all $|u| < \varepsilon$, and conclude that X is almost surely constant.

(c) Let X, Y, and Z be independent $\mathrm{N}(0,1)$ random variables. Calculate the characteristic function of $\eta = XY - Z$, given that $\phi_X(u) = e^{-u^2/2}$.

25J Applied Probability

A passenger plane with N numbered seats is about to take off; $N-1$ seats have already been taken, and now the last passenger enters the cabin. The first $N-1$ passengers were advised by the crew, rather imprudently, to take their seats completely at random, but the last passenger is determined to sit in the place indicated on his ticket. If his place is free, he takes it, and the plane is ready to fly. However, if his seat is taken, he insists that the occupier vacates it. In this case the occupier decides to follow the same rule: if the free seat is his, he takes it, otherwise he insists on his place being vacated. The same policy is then adopted by the next unfortunate passenger, and so on. Each move takes a random time which is exponentially distributed with mean μ^{-1}. What is the expected duration of the plane delay caused by these displacements?

26J Principles of Statistics

Write an essay on the rôle of the Metropolis–Hastings algorithm in computational Bayesian inference on a parametric model. You may for simplicity assume that the parameter space is finite. Your essay should:

(a) explain what problem in Bayesian inference the Metropolis–Hastings algorithm is used to tackle;

(b) fully justify that the algorithm does indeed deliver the required information about the model;

(c) discuss any implementational issues that need care.

27I Stochastic Financial Models

Let r denote the riskless rate and let $\sigma > 0$ be a fixed volatility parameter.

(a) Let $(S_t)_{t\geqslant 0}$ be a Black–Scholes asset with zero dividends:

$$S_t = S_0 \exp(\sigma B_t + (r - \sigma^2/2)t) \ ,$$

where B is standard Brownian motion. Derive the Black–Scholes partial differential equation for the price of a European option on S with bounded payoff $\varphi(S_T)$ at expiry T:

$$\partial_t V + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS} V + r\, S \partial_S V - rV = 0, \quad V(T, \cdot) = \varphi(\cdot) \ .$$

[*You may use the fact that for C^2 functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying exponential growth conditions, and standard Brownian motion B, the process*

$$C_t^f = f(t, B_t) - \int_0^t \big(\partial_t f + \tfrac{1}{2}\partial_{BB} f\big)(s, B_s)\, ds$$

is a martingale.]

(b) Indicate the changes in your argument when the asset pays dividends continuously at rate $D > 0$. Find the corresponding Black–Scholes partial differential equation.

(c) Assume $D = 0$. Find a closed form solution for the time-0 price of a European power option with payoff S_T^n.

28I Optimization and Control

A discrete-time controlled Markov process evolves according to

$$X_{t+1} = \lambda X_t + u_t + \varepsilon_t, \quad t = 0, 1, \ldots,$$

where the ε are independent zero-mean random variables with common variance σ^2, and λ is a known constant.

Consider the problem of minimizing

$$F_{t,T}(x) = \mathbb{E}\left[\sum_{j=t}^{T-1} \beta^{j-t} C(X_j, u_j) + \beta^{T-t} R(X_T)\right],$$

where $C(x,u) = \frac{1}{2}(u^2 + ax^2)$, $\beta \in (0,1)$ and $R(x) = \frac{1}{2}a_0 x^2 + b_0$. Show that the optimal control at time j takes the form $u_j = k_{T-j} X_j$ for certain constants k_i. Show also that the minimized value for $F_{t,T}(x)$ is of the form

$$\tfrac{1}{2} a_{T-t} x^2 + b_{T-t}$$

for certain constants a_j, b_j. Explain how these constants are to be calculated. Prove that the equation

$$f(z) \equiv a + \frac{\lambda^2 \beta z}{1 + \beta z} = z$$

has a unique positive solution $z = a_*$, and that the sequence $(a_j)_{j \geqslant 0}$ converges monotonically to a_*.

Prove that the sequence $(b_j)_{j \geqslant 0}$ converges, to the limit

$$b_* \equiv \frac{\beta \sigma^2 a_*}{2(1-\beta)} .$$

Finally, prove that $k_j \to k_* \equiv -\beta a_* \lambda / (1 + \beta a_*)$.

29A Partial Differential Equations

Write down the formula for the solution $u = u(t, x)$ for $t > 0$ of the initial value problem for the n-dimensional heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 \ ,$$

$$u(0, x) = g(x) \ ,$$

for $g : \mathbb{R}^n \to \mathbb{C}$ a given smooth bounded function.

State and prove the Duhamel principle giving the solution $v(t, x)$ for $t > 0$ to the inhomogeneous initial value problem

$$\frac{\partial v}{\partial t} - \Delta v = f \ ,$$

$$v(0, x) = g(x) \ ,$$

for $f = f(t, x)$ a given smooth bounded function.

For the case $n = 4$ and when $f = f(x)$ is a fixed Schwartz function (independent of t), find $v(t, x)$ and show that $w(x) = \lim_{t\to+\infty} v(t, x)$ is a solution of

$$-\Delta w = f \ .$$

[*Hint: you may use without proof the fact that the fundamental solution of the Laplacian on* $\mathbb{R}^4$ *is* $-1/(4\pi^2|x|^2)$.]

30B Asymptotic Methods

The Airy function $\mathrm{Ai}(z)$ is defined by

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_C \exp\left(-\frac{1}{3}t^3 + zt\right) \, dt \ ,$$

where the contour C begins at infinity along the ray $\arg(t) = 4\pi/3$ and ends at infinity along the ray $\arg(t) = 2\pi/3$. Restricting attention to the case where z is real and positive, use the method of steepest descent to obtain the leading term in the asymptotic expansion for $\mathrm{Ai}(z)$ as $z \to \infty$:

$$\mathrm{Ai}(z) \sim \frac{\exp\left(-\frac{2}{3}z^{3/2}\right)}{2\pi^{1/2}z^{1/4}} \ .$$

[*Hint: put* $t = z^{1/2}\tau$.]

31E Integrable Systems

The solution of the initial value problem of the KdV equation is given by

$$q(x,t) = -2i \lim_{k\to\infty} k \frac{\partial N}{\partial x}(x,t,k) \ ,$$

where the scalar function $N(x,t,k)$ can be obtained by solving the following Riemann–Hilbert problem:

$$\frac{M(x,t,k)}{a(k)} = N(x,t,-k) + \frac{b(k)}{a(k)} \exp\left(2ikx + 8ik^3 t\right) N(x,t,k), \quad k \in \mathbb{R},$$

M, N and a are the boundary values of functions of k that are analytic for $\operatorname{Im} k > 0$ and tend to unity as $k \to \infty$. The functions $a(k)$ and $b(k)$ can be determined from the initial condition $q(x,0)$.

Assume that M can be written in the form

$$\frac{M}{a} = \mathcal{M}(x,t,k) + \frac{c \exp\left(-2px + 8p^3 t\right) N(x,t,ip)}{k - ip}, \quad \operatorname{Im} k \geqslant 0,$$

where $\mathcal{M}$ as a function of k is analytic for $\operatorname{Im} k > 0$ and tends to unity as $k \to \infty$; c and p are constants and $p > 0$.

(a) By solving the above Riemann–Hilbert problem find a linear equation relating $N(x,t,k)$ and $N(x,t,ip)$.

(b) By solving this equation explicitly in the case that $b = 0$ and letting $c = 2ipe^{-2x_0}$, compute the one-soliton solution.

(c) Assume that $q(x,0)$ is such that $a(k)$ has a simple zero at $k = ip$. Discuss the dominant form of the solution as $t \to \infty$ and $x/t = O(1)$.

32D Principles of Quantum Mechanics

Consider a Hamiltonian H with known eigenstates and eigenvalues (possibly degenerate). Derive a general method for calculating the energies of a new Hamiltonian $H + \lambda V$ to first order in the parameter λ. Apply this method to find approximate expressions for the new energies close to an eigenvalue E of H, given that there are just two orthonormal eigenstates $|1\rangle$ and $|2\rangle$ corresponding to E and that

$$\langle 1|V|1\rangle = \langle 2|V|2\rangle = \alpha \ , \qquad \langle 1|V|2\rangle = \langle 2|V|1\rangle = \beta \ .$$

A charged particle of mass m moves in two-dimensional space but is confined to a square box $0 \leqslant x,\, y \leqslant a$. In the absence of any potential within this region the allowed wavefunctions are

$$\psi_{pq}(x,y) = \frac{2}{a}\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{a} \ , \qquad p,\, q = 1, 2, \ldots ,$$

inside the box, and zero outside. A weak electric field is now applied, modifying the Hamiltonian by a term $\lambda xy/a^2$, where $\lambda m a^2/\hbar^2$ is small. Show that the three lowest new energy levels for the particle are approximately

$$\frac{\hbar^2\pi^2}{ma^2} + \frac{\lambda}{4} \ , \qquad \frac{5\hbar^2\pi^2}{2ma^2} + \lambda\Big(\frac{1}{4} \pm \Big(\frac{4}{3\pi}\Big)^4\Big) \ .$$

[*It may help to recall that* $2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$.]

33A Applications of Quantum Mechanics

Consider a one-dimensional crystal of lattice space b, with atoms having positions x_s and momenta p_s, $s = 0, 1, 2, \ldots, N-1$, such that the classical Hamiltonian is

$$H = \sum_{s=0}^{N-1} \left(\frac{p_s^2}{2m} + \tfrac{1}{2} m \lambda^2 \big(x_{s+1} - x_s - b \big)^2 \right) ,$$

where we identify $x_N = x_0$. Show how this may be quantized to give the energy eigenstates consisting of a ground state $|0\rangle$ together with free phonons with energy $\hbar\omega(k_r)$ where $k_r = 2\pi r / Nb$ for suitable integers r. Obtain the following expression for the quantum operator x_s

$$x_s = s\, b + \left(\frac{\hbar}{2mN} \right)^{\frac{1}{2}} \sum_r \frac{1}{\sqrt{\omega(k_r)}} \left(a_r e^{i k_r s b} + a_r{}^\dagger e^{-i k_r s b} \right) ,$$

where $a_r, a_r{}^\dagger$ are annihilation and creation operators, respectively.

An interaction involves the matrix element

$$M = \sum_{s=0}^{N-1} \langle 0 | e^{i q x_s} | 0 \rangle .$$

Calculate this and show that $|M|^2$ has its largest value when $q = 2\pi n / b$ for integer n. Disregard the case $\omega(k_r) = 0$.

[*You may use the relations*

$$\sum_{s=0}^{N-1} e^{i k_r s b} = \begin{cases} N\,, & r = Nb\,; \\ 0 & \textit{otherwise,} \end{cases}$$

and $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ *if* $[A,B]$ *commutes with* A *and with* B.]

34D Statistical Physics

What is meant by the chemical potential of a thermodynamic system? Derive the Gibbs distribution with variable particle number N, for a system at temperature T and chemical potential μ. (You may assume that the volume does not vary.)

Consider a non-interacting gas of fermions in a box of fixed volume, at temperature T and chemical potential μ. Use the Gibbs distribution to find the mean occupation number of a one-particle quantum state of energy ε. Assuming that the density of states is $C\varepsilon^{1/2}$, for some constant C, deduce that the mean number of particles with energies between ε and $\varepsilon + d\varepsilon$ is

$$\frac{C\varepsilon^{\frac{1}{2}}d\varepsilon}{e^{(\varepsilon-\mu)/T}+1} .$$

Why can μ be identified with the Fermi energy ε_F when $T = 0$? Estimate the number of particles with energies greater than ε_F when T is small but non-zero.

35E Electrodynamics

A particle of rest mass m and charge q is moving along a trajectory $x^a(s)$, where s is the particle's proper time, in a given external electromagnetic field with 4-potential $A^a(x^c)$. Consider the action principle $\delta S = 0$ where the action is $S = \int L\, ds$ and

$$L(s, x^a, \dot{x}^a) = -m\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b} - qA_a(x^c)\dot{x}^a,$$

and variations are taken with fixed endpoints.

Show first that the action is invariant both under reparametrizations $s \to \alpha s + \beta$ where α and β are constants and also under a change of electromagnetic gauge. Next define the generalized momentum $P_a = \partial L/\partial \dot{x}^a$, and obtain the equation of motion

$$m\ddot{x}^a = qF^a{}_b\dot{x}^b, \qquad (*)$$

where the tensor $F^a{}_b$ should be defined and you may assume that $d/ds\ (\eta_{ab}\dot{x}^a\dot{x}^b) = 0$. Then verify from $(*)$ that indeed $d/ds\ (\eta_{ab}\dot{x}^a\dot{x}^b) = 0$.

How does P_a differ from the momentum p_a of an uncharged particle? Comment briefly on the principle of minimal coupling.

36B Fluid Dynamics II

Define the rate of strain tensor e_{ij} in terms of the velocity components u_i.

Write down the relation between e_{ij}, the pressure p and the stress tensor σ_{ij} in an incompressible Newtonian fluid of viscosity μ.

Prove that $2\mu e_{ij} e_{ij}$ is the local rate of dissipation per unit volume in the fluid.

Incompressible fluid of density ρ and viscosity μ occupies the semi-infinite domain $y > 0$ above a rigid plane boundary $y = 0$ that oscillates with velocity $(V \cos \omega t, 0, 0)$, where V and ω are constants. The fluid is at rest at $y = \infty$. Determine the velocity field produced by the boundary motion after any transients have decayed.

Evaluate the time-averaged rate of dissipation in the fluid, per unit area of boundary.

37C Waves

An acoustic waveguide consists of a long straight tube $z > 0$ with square cross-section $0 < x < a,\ \ 0 < y < a$ bounded by rigid walls. The sound speed of the gas in the tube is c_0. Find the dispersion relation for the propagation of sound waves along the tube. Show that for every dispersive mode there is a cut-off frequency, and determine the lowest cut-off frequency $\omega_{\min}$.

An acoustic disturbance is excited at $z = 0$ with a prescribed pressure perturbation $\tilde{p}(x, y, 0, t) = \tilde{P}(x, y) \exp(-i\omega t)$ with $\omega = \frac{1}{2}\omega_{\min}$. Find the pressure perturbation $\tilde{p}(x, y, z, t)$ at distances $z \gg a$ along the tube.

38C Numerical Analysis

(a) For the equation $y' = f(t, y)$, consider the following multistep method with s steps,

$$\sum_{i=0}^{s} \rho_i y_{n+i} = h \sum_{i=0}^{s} \sigma_i f(t_{n+i},\, y_{n+i})\ ,$$

where h is the step size and ρ_i, σ_i are specified constants with $\rho_s = 1$. Prove that this method is of order p if and only if

$$\sum_{i=0}^{s} \rho_i Q(t_{n+i}) = h \sum_{i=0}^{s} \sigma_i Q'(t_{n+i})$$

for any polynomial Q of degree $\leqslant p$. Deduce that there is no s-step method of order $2s+1$.

[*You may use the fact that, for any* a_i, b_i, *the Hermite interpolation problem*

$$Q(x_i) = a_i, \quad Q'(x_i) = b_i, \qquad i = 0, \ldots, s$$

is uniquely solvable in the space of polynomials of degree $2s+1$.]

(b) State the Dahlquist equivalence theorem regarding the convergence of a multistep method. Determine all the values of the real parameter $a \neq 0$ for which the multistep method

$$y_{n+3} + (2a-3)[y_{n+2} - y_{n+1}] - y_n = ha\,[f_{n+2} + f_{n+1}]$$

is convergent, and determine the order of convergence.

END OF PAPER