UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Wednesday 7 June 2006 9 to 12

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Prove that all binary quadratic forms of discriminant -7 are equivalent to $x^2 + xy + 2y^2$.

Determine which prime numbers p are represented by $x^2 + xy + 2y^2$.

2G Topics in Analysis

(a) State Chebyshev's equal ripple criterion.

(b) Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \cos 4\pi x \ ,$$

and let g be a polynomial of degree 7. Prove that there exists an $x \in [-1, 1]$ such that $|f(x) - g(x)| \geqslant 1$.

3F Geometry and Groups

Determine whether the following elements of $\mathrm{PSL}_2(\mathbb{R})$ are elliptic, parabolic, or hyperbolic. Justify your answers.

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix} , \qquad \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix} .$$

In the case of the first of these transformations find the fixed points.

4G Coding and Cryptography

Let Σ_1 and Σ_2 be alphabets of sizes m and a. What does it mean to say that an a-ary code $f : \Sigma_1 \to \Sigma_2^*$ is decipherable? Show that if f is decipherable then the word lengths $s_1, \ldots, s_m$ satisfy

$$\sum_{i=1}^{m} a^{-s_i} \leqslant 1 \ .$$

Find a decipherable binary code consisting of codewords 011, 0111, 01111, 11111, and three further codewords of length 2. How do you know the example you have given is decipherable?

5I Statistical Modelling

Let $Y_1, \ldots, Y_n$ be independent Poisson random variables with means $\mu_1, \ldots, \mu_n$, for $i = 1, \ldots, n$, where $\log(\mu_i) = \beta x_i$, for some known constants x_i and an unknown parameter β. Find the log-likelihood for β.

By first computing the first and second derivatives of the log-likelihood for β, explain the algorithm you would use to find the maximum likelihood estimator, $\hat{\beta}$.

6B Mathematical Biology

Two interacting populations of prey and predators, with populations $u(t)$, $v(t)$ respectively, obey the evolution equations (with all parameters positive)

$$\frac{du}{dt} = u(\mu_1 - \alpha_1 v - \delta u) \ ,$$
$$\frac{dv}{dt} = v(-\mu_2 + \alpha_2 u) - \epsilon \ .$$

Give an explanation in terms of population dynamics of each of the terms in these equations.

Show that if $\alpha_2 \mu_1 > \delta \mu_2$ there are two non-trivial fixed points with $u, v \neq 0$, provided ϵ is sufficiently small. Find the trace and determinant of the Jacobian in terms of u, v and show that, when δ and ϵ are very small, the fixed point with $u \approx \mu_1/\delta$, $v \approx \epsilon\delta/\mu_1\alpha_2$ is always unstable.

7E Dynamical Systems

Explain what is meant by a *strict Lyapunov function* on a domain $\mathcal{D}$ containing the origin for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$. Define the *domain of stability* of a fixed point $\mathbf{x}_0$.

By considering the function $V = \frac{1}{2}(x^2+y^2)$ show that the origin is an asymptotically stable fixed point of

$$\dot{x} = -2x + y + x^3 - xy^2 \ ,$$
$$\dot{y} = -x - 2y + 6x^2 y + 4y^3 \ .$$

Show also that its domain of stability includes $x^2+y^2 < \frac{1}{2}$ and is contained in $x^2+y^2 \leqslant 2$.

8E Further Complex Methods

The function $F(t)$ is defined, for $\mathrm{Re}\,t > -1$, by

$$F(t) = \int_0^\infty \frac{u^t e^{-u}}{1+u}\,du$$

and by analytic continuation elsewhere in the complex t-plane. By considering the integral of a suitable function round a Hankel contour, obtain the analytic continuation of $F(t)$ and hence show that singularities of $F(t)$ can occur only at $z = -1,\ -2,\ -3,\ \ldots$.

9C Classical Dynamics

Two point masses, each of mass m, are constrained to lie on a straight line and are connected to each other by a spring of force constant k. The left-hand mass is also connected to a wall on the left by a spring of force constant j. The right-hand mass is similarly connected to a wall on the right, by a spring of force constant ℓ, so that the potential energy is

$$V \;=\; \tfrac{1}{2}k(\eta_1-\eta_2)^2 \;+\; \tfrac{1}{2}j\eta_1^2 \;+\; \tfrac{1}{2}\ell\eta_2^2\ ,$$

where η_i is the distance from equilibrium of the i^{th} mass. Derive the equations of motion. Find the frequencies of the normal modes.

10D Cosmology

The total energy of a gas can be expressed in terms of a momentum integral

$$E = \int_0^\infty \mathcal{E}(p)\,\bar{n}(p)\;dp\ ,$$

where p is the particle momentum, $\mathcal{E}(p) = c\sqrt{p^2+m^2c^2}$ is the particle energy and $\bar{n}(p)\;dp$ is the average number of particles in the momentum range $p \to p+dp$. Consider particles in a cubic box of side L with $p \propto L^{-1}$. Explain why the momentum varies as

$$\frac{dp}{dV} = -\frac{p}{3V}\ .$$

Consider the overall change in energy dE due to the volume change dV. Given that the volume varies slowly, use the thermodynamic result $dE = -P\;dV$ (at fixed particle number N and entropy S) to find the pressure

$$P = \frac{1}{3V}\int_0^\infty p\,\mathcal{E}'(p)\,\bar{n}(p)\;dp\,.$$

Use this expression to derive the equation of state for an ultrarelativistic gas.

During the radiation-dominated era, photons remain in equilibrium with energy density $\epsilon_\gamma \propto T^4$ and number density $n_\gamma \propto T^3$. Briefly explain why the photon temperature falls inversely with the scale factor, $T \propto a^{-1}$. Discuss the implications for photon number and entropy conservation.

SECTION II

11G Topics in Analysis

(a) Let K be a closed subset of the unit disc in $\mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be a rational function with all its poles of modulus strictly greater than 1. Explain why f can be approximated uniformly on K by polynomials.
[*Standard results from complex analysis may be assumed.*]

(b) With K as above, define Λ to be the set of all $\lambda \in \mathbb{C} \setminus K$ such that the function $(z - \lambda)^{-1}$ can be uniformly approximated on K by polynomials. If $\lambda \in \Lambda$, prove that there is some $\delta > 0$ such that $\mu \in \Lambda$ whenever $|\lambda - \mu| < \delta$.

12G Coding and Cryptography

Define a cyclic code. Show that there is a bijection between the cyclic codes of length n, and the factors of $X^n - 1$ in $\mathbb{F}_2[X]$.

If n is an odd integer then we can find a finite extension K of $\mathbb{F}_2$ that contains a primitive nth root of unity α. Show that a cyclic code of length n with defining set $\{\alpha, \alpha^2, \ldots, \alpha^{\delta-1}\}$ has minimum distance at least δ. Show that if $n = 7$ and $\delta = 3$ then we obtain Hamming's original code.

[*You may quote a formula for the Vandermonde determinant without proof.*]

13B Mathematical Biology

Consider the discrete predator-prey model for two populations N_t, P_t of prey and predators, respectively:

$$\left.\begin{aligned} N_{t+T} &= rN_t \exp(-aP_t) \\ P_{t+T} &= sN_t(1 - b\exp(-aP_t)) \end{aligned}\right\}, \qquad (*)$$

where r, s, a, b are constants, all assumed to be positive.

(a) Give plausible explanations of the meanings of T, r, s, a, b.

(b) Nondimensionalize equations $(*)$ to show that with appropriate rescaling they may be reduced to the form

$$\left.\begin{aligned} n_{t+1} &= rn_t \exp(-p_t) \\ p_{t+1} &= n_t(1 - b\exp(-p_t)) \end{aligned}\right\}.$$

(c) Now assume that $b < 1$, $r > 1$. Show that the origin is unstable, and that there is a nontrivial fixed point $(n, p) = (n_c(b, r), p_c(b, r))$. Investigate the stability of this point by writing $(n_t, p_t) = (n_c + n_t', p_c + p_t')$ and linearizing. Express the linearized equations as a second order recurrence relation for n_t', and hence show that n_t' satisfies an equation of the form

$$n_t' = A\lambda_1^t + B\lambda_2^t$$

where the quantities $\lambda_{1,2}$ satisfy $\lambda_1 + \lambda_2 = 1 + bn_c/r$, $\lambda_1\lambda_2 = n_c$ and A, B are constants. Give a similar expression for p_t' for the same values of A, B.

Show that when r is just greater than unity the λ_i $(i = 1, 2)$ are real and both less than unity, while if n_c is just greater than unity then the λ_i are complex with modulus greater than one. Show also that n_c increases monotonically with r and that if the roots are real neither of them can be unity.

Deduce that the fixed point is stable for sufficiently small r but loses stability for a value of r that depends on b but is certainly less than $e = \exp(1)$. Give an equation that determines the value of r where stability is lost, and an equation that gives the argument of the eigenvalue at this point. Sketch the behaviour of the moduli of the eigenvalues as functions of r.

14E Dynamical Systems

Let $F : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. Explain what is meant by saying (a) that F has a *horseshoe*, (b) that F is *chaotic* (Glendinning's definition).

Consider the tent map defined on the interval $[0,1]$ by

$$F(x) = \begin{cases} \mu x & 0 \leqslant x < \frac{1}{2} \\ \mu(1-x) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

with $1 < \mu \leqslant 2$.

Find the non-zero fixed point x_0 and the points $x_{-1} < \frac{1}{2} < x_{-2}$ that satisfy

$$F^2(x_{-2}) = F(x_{-1}) = x_0 \; .$$

Sketch a graph of F and F^2 showing the points corresponding to x_{-2}, x_{-1} and x_0. Hence show that F^2 has a horseshoe if $\mu \geqslant 2^{1/2}$.

Explain briefly why F^4 has a horseshoe when $2^{1/4} \leqslant \mu < 2^{1/2}$ and why there are periodic points arbitrarily close to x_0 for $\mu \geqslant 2^{1/2}$, but no such points for $2^{1/4} \leqslant \mu < 2^{1/2}$.

15D Cosmology

(a) Consider a homogeneous and isotropic universe filled with relativistic matter of mass density $\rho(t)$ and scale factor $a(t)$. Consider the energy $E(t) \equiv \rho(t)c^2 V(t)$ of a small fluid element in a comoving volume V_0 where $V(t) = a^3(t)V_0$. Show that for slow (adiabatic) changes in volume, the density will satisfy the fluid conservation equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P/c^2\right) ,$$

where P is the pressure.

(b) Suppose that a flat ($k = 0$) universe is filled with two matter components:

(i) radiation with an equation of state $P_{\rm r} = \frac{1}{3}\rho_{\rm r} c^2$.

(ii) a gas of cosmic strings with an equation of state $P_{\rm s} = -\frac{1}{3}\rho_{\rm s} c^2$.

Use the fluid conservation equation to show that the total relativistic mass density behaves as

$$\rho \;=\; \frac{\rho_{\rm r0}}{a^4} + \frac{\rho_{\rm s0}}{a^2} ,$$

where $\rho_{\rm r0}$ and $\rho_{\rm s0}$ are respectively the radiation and string densities today (that is, at $t = t_0$ when $a(t_0) = 1$). Assuming that both the Hubble parameter today H_0 and the ratio $\beta \equiv \rho_{\rm r0}/\rho_{\rm s0}$ are known, show that the Friedmann equation can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{a^4}\left(\frac{a^2 + \beta}{1 + \beta}\right) .$$

Solve this equation to find the following solution for the scale factor

$$a(t) = \frac{(H_0 t)^{1/2}}{(1+\beta)^{1/2}}\left[H_0 t + 2\beta^{1/2}(1+\beta)^{1/2}\right]^{1/2} .$$

Show that the scale factor has the expected asymptotic behaviour at early times $t \to 0$.

Hence show that the age of this universe today is

$$t_0 \;=\; H_0^{-1}(1+\beta)^{1/2}\left[(1+\beta)^{1/2} - \beta^{1/2}\right] ,$$

and that the time $t_{\rm eq}$ of equal radiation and string densities ($\rho_{\rm r} = \rho_{\rm s}$) is

$$t_{\rm eq} \;=\; H_0^{-1}\left(\sqrt{2} - 1\right)\beta^{1/2}(1+\beta)^{1/2} .$$

16H Logic and Set Theory

Which of the following statements are true, and which false? Justify your answers.

(a) For any ordinals α and β with $\beta \neq 0$, there exist ordinals γ and δ with $\delta < \beta$ such that $\alpha = \beta.\gamma + \delta$.

(b) For any ordinals α and β with $\beta \neq 0$, there exist ordinals γ and δ with $\delta < \beta$ such that $\alpha = \gamma.\beta + \delta$.

(c) $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$ for all α, β, γ.

(d) $(\alpha + \beta).\gamma = \alpha.\gamma + \beta.\gamma$ for all α, β, γ.

(e) Any ordinal of the form $\omega.\alpha$ is a limit ordinal.

(f) Any limit ordinal is of the form $\omega.\alpha$.

17F Graph Theory

Let G be a bipartite graph with vertex classes X and Y. State Hall's necessary condition for G to have a matching from X to Y, and prove that it is sufficient.

Deduce a necessary and sufficient condition for G to have $|X| - d$ independent edges, where d is a natural number.

Show that the maximum size of a set of independent edges in G is equal to the minimum size of a subset $S \subset V(G)$ such that every edge of G has an end vertex in S.

18H Galois Theory

Write an essay on ruler and compass construction.

19F Representation Theory

(a) Let G be S_4, the symmetric group on four letters. Determine the character table of G.
[*Begin by listing the conjugacy classes and their orders.*]

(b) For each irreducible representation V of $G = S_4$, decompose $\mathrm{Res}_{A_4}^{S_4}(V)$ into irreducible representations. You must justify your answer.

20G Number Fields

Let $K = \mathbb{Q}(\sqrt{26})$ and let $\varepsilon = 5 + \sqrt{26}$. By Dedekind's theorem, or otherwise, show that the ideal equations

$$2 = [2, \varepsilon + 1]^2, \quad 5 = [5, \varepsilon + 1][5, \varepsilon - 1], \quad [\varepsilon + 1] = [2, \varepsilon + 1][5, \varepsilon + 1]$$

hold in K. Deduce that K has class number 2.

Show that ε is the fundamental unit in K. Hence verify that all solutions in integers x, y of the equation $x^2 - 26y^2 = \pm 10$ are given by

$$x + \sqrt{26}y = \pm \varepsilon^n (\varepsilon \pm 1) \qquad (n = 0, \pm 1, \pm 2, \ldots)\ .$$

[*It may be assumed that the Minkowski constant for K is $\frac{1}{2}$.*]

21H Algebraic Topology

State the simplicial approximation theorem. Compute the number of 0-simplices (vertices) in the barycentric subdivision of an n-simplex and also compute the number of n-simplices. Finally, show that there are at most countably many homotopy classes of continuous maps from the 2-sphere to itself.

22G Linear Analysis

Let X be a metric space. Define what it means for a subset $E \subset X$ to be of *first* or *second category*. State and prove a version of the Baire category theorem. For $1 \leqslant p \leqslant \infty$, show that the set ℓ_p is of first category in the normed space ℓ_r when $r > p$ and ℓ_r is given its standard norm. What about $r = p$?

23F Riemann Surfaces

Define the terms *Riemann surface*, *holomorphic map* between Riemann surfaces, and *biholomorphic map*.

(a) Prove that if two holomorphic maps f, g coincide on a non-empty open subset of a connected Riemann surface R then $f = g$ everywhere on R.

(b) Prove that if $f : R \to S$ is a non-constant holomorphic map between Riemann surfaces and $p \in R$ then there is a choice of co-ordinate charts ϕ near p and ψ near $f(p)$, such that $(\psi \circ f \circ \phi^{-1})(z) = z^n$, for some non-negative integer n. Deduce that a holomorphic bijective map between Riemann surfaces is biholomorphic.

[*The inverse function theorem for holomorphic functions on open domains in $\mathbb{C}$ may be used without proof if accurately stated.*]

24H Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface.

(a) Define the exponential map $\exp_p$ at a point $p \in S$. Assuming that $\exp_p$ is smooth, show that $\exp_p$ is a diffeomorphism in a neighbourhood of the origin in T_pS.

(b) Given a parametrization around $p \in S$, define the Christoffel symbols and show that they only depend on the coefficients of the first fundamental form.

(c) Consider a system of normal co-ordinates centred at p, that is, Cartesian co-ordinates (x, y) in T_pS and parametrization given by $(x, y) \mapsto \exp_p(xe_1 + ye_2)$, where $\{e_1, e_2\}$ is an orthonormal basis of T_pS. Show that all of the Christoffel symbols are zero at p.

25J Probability and Measure

(a) What is meant by saying that $(\Omega, \mathcal{A}, \mu)$ is a measure space? Your answer should include clear definitions of any terms used.

(b) Consider the following sequence of Borel-measurable functions on the measure space $(\mathbb{R}, \mathcal{L}, \lambda)$, with the Lebesgue σ-algebra $\mathcal{L}$ and Lebesgue measure λ:

$$f_n(x) = \begin{cases} 1/n & \text{if } 0 \leqslant x \leqslant e^n; \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } \ n \in \mathbb{N} \ .$$

For each $p \in [1, \infty]$, decide whether the sequence $(f_n)_{n \in \mathbb{N}}$ converges in L^p as $n \to \infty$.
Does $(f_n)_{n \in \mathbb{N}}$ converge almost everywhere?
Does $(f_n)_{n \in \mathbb{N}}$ converge in measure?
Justify your answers.

For parts (c) and (d), let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real-valued, Borel-measurable functions on a probability space $(\Omega, \mathcal{A}, \mu)$.

(c) Prove that $\{x \in \Omega : f_n(x) \text{ converges to a finite limit}\} \in \mathcal{A}$.

(d) Show that $f_n \to 0$ almost surely if and only if $\sup_{m \geqslant n} |f_m| \to 0$ in probability.

26J Applied Probability

(a) Define a renewal process (X_t) with independent, identically-distributed holding times S_1, S_2, ... State without proof the strong law of large numbers for (X_t). State without proof the elementary renewal theorem for the mean value $m(t) = \mathbb{E}X_t$.

(b) A circular bus route consists of ten bus stops. At exactly 5am, the bus starts letting passengers in at the main bus station (stop 1). It then proceeds to stop 2 where it stops to let passengers in and out. It continues in this fashion, stopping at stops 3 to 10 in sequence. After leaving stop 10, the bus heads to stop 1 and the cycle repeats. The travel times between stops are exponentially distributed with mean 4 minutes, and the time required to let passengers in and out at each stop are exponentially distributed with mean 1 minute. Calculate approximately the average number of times the bus has gone round its route by 1pm.

When the driver's shift finishes, at exactly 1pm, he immediately throws all the passengers off the bus if the bus is already stopped, or otherwise, he drives to the next stop and then throws the passengers off. He then drives as fast as he can round the rest of the route to the main bus station. Giving reasons but not proofs, calculate approximately the average number of stops he will drive past at the end of his shift while on his way back to the main bus station, not including either the stop at which he throws off the passengers or the station itself.

27J Principles of Statistics

Let $\{f(\cdot|\theta) : \theta \in \Theta\}$ be a parametric family of densities for observation X. What does it mean to say that the statistic $T \equiv T(X)$ is *sufficient* for θ? What does it mean to say that T is *minimal sufficient*?

State the Rao–Blackwell theorem. State the Cramér–Rao lower bound for the variance of an unbiased estimator of a (scalar) parameter, taking care to specify any assumptions needed.

Let $X_1, \ldots, X_n$ be a sample from a $U(0, \theta)$ distribution, where the positive parameter θ is unknown. Find a minimal sufficient statistic T for θ. If $h(T)$ is an unbiased estimator for θ, find the form of h, and deduce that this estimator is minimum-variance unbiased. Would it be possible to reach this conclusion using the Cramér–Rao lower bound?

28I Stochastic Financial Models

(a) In the context of a single-period financial market with n traded assets and a single riskless asset earning interest at rate r, what is an arbitrage? What is an equivalent martingale measure? Explain marginal utility pricing, and how it leads to an equivalent martingale measure.

(b) Consider the following single-period market with two assets. The first is a riskless bond, worth 1 at time 0, and 1 at time 1. The second is a share, worth 1 at time 0 and worth S_1 at time 1, where S_1 is uniformly distributed on the interval $[0, a]$, where $a > 0$. Under what condition on a is this model arbitrage free? When it is, characterise the set $\mathcal{E}$ of equivalent martingale measures.

An agent with C^2 utility U and with wealth w at time 0 aims to pick the number θ of shares to hold so as to maximise his expected utility of wealth at time 1. Show that he will choose θ to be positive if and only if $a > 2$.

An option pays $(S_1 - 1)^+$ at time 1. Assuming that $a = 2$, deduce that the agent's price for this option will be $1/4$, and show that the range of possible prices for this option as the pricing measure varies in $\mathcal{E}$ is the interval $(0, \frac{1}{2})$.

29I Optimization and Control

A policy π is to be chosen to maximize

$$F(\pi, x) = \mathbb{E}_\pi \left[\sum_{t \geqslant 0} \beta^t r(x_t, u_t) \,\middle|\, x_0 = x \right],$$

where $0 < \beta \leqslant 1$. Assuming that $r \geqslant 0$, prove that π is optimal if $F(\pi, x)$ satisfies the optimality equation.

An investor receives at time t an income of x_t of which he spends u_t, subject to $0 \leqslant u_t \leqslant x_t$. The reward is $r(x_t, u_t) = u_t$, and his income evolves as

$$x_{t+1} = x_t + (x_t - u_t)\varepsilon_t,$$

where $(\varepsilon_t)_{t \geqslant 0}$ is a sequence of independent random variables with common mean $\theta > 0$. If $0 < \beta \leqslant 1/(1+\theta)$, show that the optimal policy is to take $u_t = x_t$ for all t.

What can you say about the problem if $\beta > 1/(1+\theta)$?

30A Partial Differential Equations

Define a *fundamental solution* of a constant-coefficient linear partial differential operator, and prove that the distribution defined by the function $N : \mathbb{R}^3 \to \mathbb{R}$

$$N(x) = (4\pi|x|)^{-1}$$

is a fundamental solution of the operator $-\Delta$ on $\mathbb{R}^3$.

State and prove the mean value property for harmonic functions on $\mathbb{R}^3$ and deduce that any two smooth solutions of

$$-\Delta u = f \ , \qquad f \in C^\infty(\mathbb{R}^3)$$

which satisfy the condition

$$\lim_{|x|\to\infty} u(x) = 0$$

are in fact equal.

31E Integrable Systems

Let $\phi(t)$ satisfy the singular integral equation

$$\left(t^4 + t^3 - t^2\right) \frac{\phi(t)}{2} + \frac{(t^4 - t^3 - t^2)}{2\pi i} \oint_C \frac{\phi(\tau)}{\tau - t} \, d\tau = (A-1)t^3 + t^2 \ ,$$

where C denotes the circle of radius 2 centred on the origin, $\oint$ denotes the principal value integral and A is a constant. Derive the associated Riemann–Hilbert problem, and compute the canonical solution of the corresponding homogeneous problem.

Find the value of A such that $\phi(t)$ exists, and compute the unique solution $\phi(t)$ if A takes this value.

32A Principles of Quantum Mechanics

Let $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ denote the eigenstates of S_z for a particle of spin $\frac{1}{2}$. Show that

$$|{\uparrow}\theta\rangle = \cos\frac{\theta}{2}\,|{\uparrow}\rangle + \sin\frac{\theta}{2}\,|{\downarrow}\rangle\ , \qquad |{\downarrow}\theta\rangle = -\sin\frac{\theta}{2}\,|{\uparrow}\rangle + \cos\frac{\theta}{2}\,|{\downarrow}\rangle$$

are eigenstates of $S_z\cos\theta + S_x\sin\theta$ for any θ. Show also that the composite state

$$|\chi\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle\,|{\downarrow}\rangle \ - \ |{\downarrow}\rangle\,|{\uparrow}\rangle\right)\ ,$$

for two spin-$\frac{1}{2}$ particles, is unchanged under a transformation

$$|{\uparrow}\rangle \ \mapsto \ |{\uparrow}\theta\rangle\ , \qquad |{\downarrow}\rangle \ \mapsto \ |{\downarrow}\theta\rangle \qquad (*)$$

applied to all one-particle states. Hence, by considering the action of certain components of the spin operator for the composite system, show that $|\chi\rangle$ is a state of total spin zero.

Two spin-$\frac{1}{2}$ particles A and B have combined spin zero (as in the state $|\chi\rangle$ above) but are widely separated in space. A magnetic field is applied to particle B in such a way that its spin states are transformed according to $(*)$, for a certain value of θ, while the spin states of particle A are unaffected. Once this has been done, a measurement is made of S_z for particle A, followed by a measurement of S_z for particle B. List the possible results for this pair of measurements and find the total probability, in terms of θ, for each pair of outcomes to occur. For which outcomes is the two-particle system left in an eigenstate of the combined total spin operator, S^2, and what is the eigenvalue for each such outcome?

$$\left[\qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \right]$$

33D Applications of Quantum Mechanics

State and prove Bloch's theorem for the electron wave functions for a periodic potential $V(\mathbf{r}) = V(\mathbf{r}+\mathbf{l})$ where $\mathbf{l} = \sum_i n_i\, \mathbf{a}_i$ is a lattice vector.

What is the reciprocal lattice? Explain why the Bloch wave-vector $\mathbf{k}$ is arbitrary up to $\mathbf{k} \to \mathbf{k}+\mathbf{g}$, where $\mathbf{g}$ is a reciprocal lattice vector.

Describe in outline why one can expect energy bands $E_n(\mathbf{k}) = E_n(\mathbf{k}+\mathbf{g})$. Explain how $\mathbf{k}$ may be restricted to a Brillouin zone B and show that the number of states in volume d^3k is

$$\frac{2}{(2\pi)^3}\, \mathrm{d}^3 k\,.$$

Assuming that the velocity of an electron in the energy band with Bloch wave-vector $\mathbf{k}$ is

$$\mathbf{v}(\mathbf{k}) = \frac{1}{\hbar}\frac{\partial}{\partial \mathbf{k}} E_n(\mathbf{k})\,,$$

show that the contribution to the electric current from a full energy band is zero. Given that $n(\mathbf{k}) = 1$ for each occupied energy level, show that the contribution to the current density is then

$$\mathbf{j} = -e\,\frac{2}{(2\pi)^3}\int_B \mathrm{d}^3 k\; n(\mathbf{k})\mathbf{v}(\mathbf{k})\,,$$

where $-e$ is the electron charge.

34D Statistical Physics

What is meant by the heat capacity C_V of a thermodynamic system? By establishing a suitable Maxwell identity, show that

$$\left.\frac{\partial C_V}{\partial V}\right|_T = T\left.\frac{\partial^2 P}{\partial T^2}\right|_V\,. \qquad (*)$$

In a certain model of N interacting particles in a volume V and at temperature T, the partition function is

$$Z = \frac{1}{N!}(V-aN)^N (bT)^{3N/2}\,,$$

where a and b are constants. Find the equation of state and the entropy for this gas of particles. Find the energy and hence the heat capacity C_V of the gas, and verify that the relation $(*)$ is satisfied.

35A General Relativity

The Schwarzschild metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) - \left(1 - \frac{2M}{r}\right) dt^2 \ .$$

Writing $u = 1/r$, obtain the equation

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2 \ , \qquad (*)$$

determining the spatial orbit of a null (massless) particle moving in the equatorial plane $\theta = \pi/2$.

Verify that two solutions of $(*)$ are

$$\text{(i)} \qquad u = \frac{1}{3M} \ , \quad \text{and}$$

$$\text{(ii)} \qquad u = \frac{1}{3M} - \frac{1}{M} \frac{1}{\cosh\phi + 1} \ .$$

What is the significance of solution (i)? Sketch solution (ii) and describe its relation to solution (i).

Show that, near $\phi = \cosh^{-1} 2$, one may approximate the solution (ii) by

$$r \sin(\phi - \cosh^{-1} 2) \approx \sqrt{27} M \ ,$$

and hence obtain the impact parameter.

36B Fluid Dynamics II

A very long cylinder of radius a translates steadily at speed V in a direction perpendicular to its axis and parallel to a plane boundary. The centre of the cylinder remains a distance $a+b$ above the plane, where $b \ll a$, and the motion takes place through an incompressible fluid of viscosity μ.

Consider the force F per unit length parallel to the plane that must be applied to the cylinder to maintain the motion. Explain why F scales according to $F \propto \mu V (a/b)^{1/2}$.

Approximating the lower cylindrical surface by a parabola, or otherwise, determine the velocity and pressure gradient fields in the space between the cylinder and the plane. Hence, by considering the shear stress on the plane, or otherwise, calculate F explicitly.

[*You may use*

$$\int_{-\infty}^{\infty}(1+x^2)^{-1}\,dx=\pi\ ,\quad \int_{-\infty}^{\infty}(1+x^2)^{-2}\,dx=\tfrac{1}{2}\pi \quad \textit{and} \quad \int_{-\infty}^{\infty}(1+x^2)^{-3}\,dx=\tfrac{3}{8}\pi\ .\Big]$$

37C Waves

The dispersion relation for waves in deep water is

$$\omega^2 = g|k|\ .$$

At time $t=0$ the water is at rest and the elevation of its free surface is $\zeta=\zeta_0\exp(-|x|/b)$ where b is a positive constant. Use Fourier analysis to find an integral expression for $\zeta(x,t)$ when $t>0$.

Use the method of stationary phase to find $\zeta(Vt,t)$ for fixed $V>0$ and $t\to\infty$.

$$\left[\int_{-\infty}^{\infty}\exp\left(ikx-\frac{|x|}{b}\right)\,dx \;=\; \frac{2b}{1+k^2b^2}\ ;\quad \int_{-\infty}^{\infty}\exp(-ax^2)\,dx \;=\; \sqrt{\frac{\pi}{a}}\quad (\mathrm{Re}\,a\geqslant 0)\ .\right]$$

38C Numerical Analysis

In the unit square the Poisson equation $\nabla^2 u = f$, with zero Dirichlet boundary conditions, is being solved by the five-point formula using a square grid of mesh size $h=1/(M+1)$,

$$u_{i,j-1}+u_{i,j+1}+u_{i-1,j}+u_{i+1,j}-4u_{i,j}=h^2 f_{i,j}\ .$$

Let $u(x,y)$ be the exact solution, and let $e_{i,j}=u_{i,j}-u(ih,jh)$ be the error of the five-point formula at the (i,j)th grid point. Justifying each step, prove that

$$\left[\sum_{i,j=1}^{M}|e_{i,j}|^2\right]^{1/2} \leqslant ch, \quad h\to 0\ ,$$

where c is some constant.

END OF PAPER